Exact Solutions to the Sine-Gordon Equation

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Introduction

Many 1-D nonlinear evolution equations allow explicit solutions that can be obtained by the **I**nverse **S**cattering **T**ransform (IST).

In this talk we consider the sine-Gordon equation, i.e.

$$u_{xt} = \sin u \,,$$

where u represents a real scalar function, $(x,\, t) \in \mathbb{R}^2$ and the subscripts denote the appropriate partial derivatives.

Sine-Gordon Equation

This equation arises in many applicative physical contexts as:

- Description of surfaces of constant negative Gaussian curvature;
- Magnetic flux propagation in Josephson junctions, i.e. gaps between two superconductors;
- Propagation of deformations along the DNA double helix.

In 1973 Ablowitz, Kaup, Newell and Segur proved that the following initial value problem for the sine-Gordon equation

$$\begin{cases} u_{xt} = \sin u, \\ u(x,0) \end{cases}$$

is solvable by using the IST method.

In order to get the solutions of this equation, it is also possible to apply other methods such as Hirota method or Bäcklund transformations.

Sine-Gordon Equation

In this talk, by applying the IST and the Marchenko method, we get explicit solutions to the sine Gordon equation.
Representing the kernels of the Marchenko equation in a separated form by using a suitable triplet of constant real matrices (A, B, C), we explicitly solve the Marchenko equations by separation of variables. The solution of the sine-Gordon equation is related (by a simple algebraic relation) to the Marchenko equation.

The main result is as follows:
Using a "suitable" triplet of constant real matrices (A, B, C) where A, B, C have dimensions $p \times p$, $p \times 1$ and $1 \times p$, we obtain this solution formula:

$$u(x,t) = -2B^\dagger F^{-1}(x,t)C^\dagger \,,$$

with $F(x,t) = e^{\beta^\dagger} + Q\, e^{-\beta} N$, where $\beta = 2Ax + \frac{1}{2} A^{-1} t$ and

$$Q = \int_0^\infty ds\, e^{-A^\dagger s} C^\dagger C e^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar} B B^\dagger e^{-A^\dagger r} \,.$$

Advantages

This procedure has several advantages:

1. It is generalizable to other integrable nonlinear evolution equations (NLS, KdV, mKdV).
2. The explicit formula found is expressed in a concise form in terms of the triplet (A, B, C). Using computer algebra, we can "unzip" the solution in terms of exponential, trigonometric, and polynomial functions of x and t. Even for matrices A of moderate order, this unzipped expression may take several pages!
3. Our solution formula contains, as special cases, many of the solutions already known in the literature (**one-soliton solution, breather solutions**). However, our solution formula allows us to construct also a new class of solutions (**multipole solutions**).
4. Choosing different triplets as input in our formula, we get a set of solutions to the sine-Gordon equation which can be used for "validation" of numerical methods.

sine-Gordon and IST

Let us consider the initial vaue problem for the sine-Gordon equation

$$\begin{cases} u_{xt} = \sin u, \\ u(x,0) = q(x) \end{cases}$$

where $u = u(x,t)$ is a real valued function.

In order to solve this problem by using the inverse scattering transform (IST) method, it is necessary to build the direct and inverse scattering for the so-called Zakharov-Shabat system

$$\begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{1}{2}\, q_x(x)\, \eta, \\ \dfrac{d\eta}{dx} = \dfrac{1}{2}\, q_x(x)\, \xi + i\lambda\eta, \end{cases}$$

where λ is the spectral parameter, $x \in \mathbb{R}$, and we assume that the potential $q_x(x)$ is in $L^1(\mathbb{R})$.

Scheme of the IST

The following diagram illustrates the IST scheme:

$$
\begin{array}{ccccc}
q(x) & \longrightarrow & q_x(x) & \xrightarrow{\text{direct scattering at } t=0} & \{R(\lambda,0),\{\lambda_j, c_j\}\} \\
\Big\downarrow \text{sine-Gordon solution} & & & & \Big\downarrow \text{time evolution} \\
q(x,t) & \longleftarrow & q_x(x,t) & \xleftarrow[\text{inverse scattering at } t]{} & \{R(\lambda,t),\{\lambda_j, c_j\, e^{-it/(2\lambda_j)}\}\}
\end{array}
$$

where $q(x,t)$ satisfy the Sine-Gordon equation, i.e. $u(x,t) = q(x,t)$.

The Scattering matrix

We only remark that from the potential $q_x(x)$ we construct the scattering matrix as follows

$$\mathbf{S}(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix}.$$

The scattering matrix is J-unitary, i.e., $S(\lambda)^\dagger J S(\lambda) = S(\lambda) J S(\lambda)^\dagger = J$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Inverse Scattering of the Zakharov-Shabat System

In order to (re)-construct the potential we have the following procedure:

1. Given the scattering data $\{R(\lambda, t),\, \lambda_j,\, \Gamma_{js}(t)\}$, we build the following integral kernel

$$\Omega(y,t) = \hat{R}(y,t) + \sum_{j=1}^{N} \sum_{s=1}^{n_j} \frac{1}{(s-1)!} y^{s-1} \Gamma_{js}(t)\, e^{i\lambda_j y}.$$

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2. Using the kernel $\Omega(y)$, we can consider the following Marchenko integral equation
$$K(x,y,t) - \Omega(y+x,t)^{\dagger} + \int_x^{\infty} dv \int_x^{\infty} dr\, K(x,v,t)\Omega(v+r,t)\Omega(r+y,t)^{\dagger} = 0.$$

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3. The potential $q(x,t)$ is connected to the above equation by means of the following relationship
$$u(x,t) = -4\int_x^\infty dr\, K(r,r,t) \,\text{or, equivalently,}\; u_x(x,t) = 4K(x,x,t).$$

Kernel of the Marchenko Equations

It is well known that the kernel of the Marchenko equation has to satisfy the following first order PDE:

$$\Omega_{yt}(y;t) = \frac{1}{2}\Omega(y;t)\,.$$

This suggests to take the kernel above introduced as (reflectionless case)

$$\Omega(y,t) = C\,e^{-yA-A^{-1}t/2}B,$$

where A is a $p \times p$ real constant matrix, B, C are, respectively, $p \times 1$ and $1 \times p$ real constant matrices, the blue factor is the so-called time factor.

Let us also assume that all the eigenvalues of A have positive real parts and (A, B, C) is a minimal triplet, i.e.,

$$\bigcap_{r=1}^{+\infty} \ker CA^{r-1} = \bigcap_{r=1}^{+\infty} \ker B^{\dagger}(A^{\dagger})^{r-1} = \{0\}.$$

It is noteworthy that the triplet yielding a minimal realization is unique up to a similarity transformation $(A, B, C) \to (EAE^{-1}, EB, CE^{-1})$ for some unique matrix E.

Two different approaches at the Marchenko integral equation

Recalling that $u(x,t)$ is a real valued function, even the kernel $\Omega(y,t)$ and the solution $K(x,y,t)$ of the Marchenko integral equation are real valued. But we can also observe that the kernel $\Omega(y,t)$ constructed via the triplet (A,B,C) is a scalar function implying that $\Omega(y,t)^\dagger = \Omega(y,t)$.

In order to solve the Marchenko equation we can proceed in two different ways:

- Approach 1: we can substitute the kernel $\Omega(y,t) = C\, e^{-yA-A^{-1}t/2}B$ in the Marchenko equation and consider it as a complex function;
- Approach 2: we take into account that $\Omega(y,t)^\dagger = \Omega(y,t)$ from the beginning when we start to solve the corresponding Marchenko equation.

Explicit Expression for the Potential: Approach 1

Putting

$$\Omega(y,t) = C\,e^{-yA-A^{-1}t/2}B, \quad \Omega(y,t)^\dagger = B^\dagger e^{-A^\dagger y-(A^\dagger)^{-1}t/2}C^\dagger,$$

into the Marchenko equation we obtain

$$K(x,y,t) - B^\dagger e^{-A^\dagger(x+y)-(A^\dagger)^{-1}t/2}C^\dagger +$$
$$\int_x^\infty dv \int_x^\infty dr\, K(x,v,t)C\,e^{-(v+r)A-A^{-1}t/2}BB^\dagger e^{-A^\dagger(r+y)-(A^\dagger)^{-1}t/2}C^\dagger = 0.$$

Defining

$$Q = \int_0^\infty d\gamma\, e^{-\gamma A^\dagger}C^\dagger C e^{-\gamma A}, \qquad N = \int_0^\infty dy\, e^{-yA}BB^\dagger e^{-yA^\dagger},$$

and looking for a solution in the following form

$$K(x,y,t) = H(x,t)\,e^{-A^\dagger y-(A^\dagger)^{-1}t/2}C^\dagger,$$

we arrive, after some easy and straightforward calculations, at the solution

$$K(x,y,t) = B^\dagger F(x,t)^{-1}e^{-A^\dagger(y-x)}C^\dagger.$$

Explicit Expression for the Potential: Approach 1

The matrix $F(x,t)$ which appear in the solution of the Marchenko equation is defined as

$$F(x,t) = e^{\beta^\dagger} + Q\, e^{-\beta} N,$$

where $\beta = 2Ax + \dfrac{1}{2}\, A^{-1} t$.

Moreover, the matrix $F(x,t)$ is invertible on the entire xt-plane and its inverse satisfies the following property: $F(x,t)^{-1} \to 0$ exponentially as $x \to \pm\infty$ for each fixed t.

Recalling the relationship between the solution of the Marchenko equation and the potential we get

$$u(x,t) = -4 \int_x^\infty dr\, B^\dagger F(r,t)^{-1} C^\dagger = -4 \int_x^\infty dr\, C[F(r,t)^\dagger]^{-1} B,$$

where we have employed the fact that $u(x,t)$ is a real value function and $\Omega(y,t)^\dagger = \Omega(y,t)$ only at the end. Using the properties of the function $F(x,t)$ we see that $u(x,t)$ given above is well defined on the entire xt-plane.

Explicit Expression for the Potential: Approach 2

Using $\Omega(y,t)^\dagger = \Omega(y,t) = C\, e^{-yA-A^{-1}t/2}B$, the Marchenko equation becomes

$$K(x,y,t) - Ce^{-A(x+y)-(A)^{-1}t/2}B + \int_x^\infty dv \int_x^\infty dr\, K(x,v,t) C\, e^{-(v+r)A-A^{-1}t/2} BCe^{-A(r+y)-(A)^{-1}t/2}B = 0.$$

Defining

$$P = \int_0^\infty d\gamma\, e^{-\gamma A} BCe^{-\gamma A}$$

and looking for a solution in the following form

$$K(x,y,t) = H_1(x,t)\, Ce^{-Ay-(A)^{-1}t/2}B,$$

we can solve the Marchenko equation in a similar way as seen in Approach 1 obtaining

$$u(x,t) = -4 \int_x^\infty dr\, CE(r,t)^{-1}B,$$

where

$$F(x,t)^\dagger = E(x,t) := e^{\beta} + P\, e^{-\beta} P, \quad \text{with } \beta = 2Ax + \frac{1}{2} A^{-1} t$$

Construction of the Explicit Solution

In order to obtain explicit solutions for the sine-Gordon equation, we can proceed, in a more direct way, as indicated below. (In blue the steps of the procedure corresponding to APPROACH 1, in red those corresponding to APPROACH 2)

- Consider a **real** triplet (A, B, C) corresponding a minimal realization for $\Omega(y, t)$ (with A is a $p \times p$, B is a $p \times m$ and C is an $n \times p$ matrix) and such that all the eigenvalues of A have positive real parts.
- Construct the constant $p \times p$ matrices Q and N that are the unique solutions, respectively, to the Lyapunov equations

$$QA + A^{\dagger} Q = C^{\dagger} C, \; AN + NA^{\dagger} = BB^{\dagger}.$$

Equivalently, construct the unique solution of the Sylvester equation

$$AP + PA = BC.$$

Lyapunov equation

Assume that the **real** triplet (A, B, C) corresponds to a minimal realization of the kernel Ω and all the eigenvalues of A have positive real parts.. Then:

1. The Lyapunov equations

$$QA + A^\dagger Q = C^\dagger C, \; AN + NA^\dagger = BB^\dagger.$$

   are uniquely solvable. We call this solutions Q and N being

$$Q = \int_0^\infty d\gamma \, e^{-\gamma A^\dagger} C^\dagger C e^{-\gamma A}, \qquad N = \int_0^\infty dy \, e^{-yA} BB^\dagger e^{-yA^\dagger},$$

   i.e. the same representation that we have found developing the IST!
2. The constant matrices Q and N are selfadjoint; i.e. $Q = Q^\dagger$, $N = N^\dagger$.
3. The constant matrices Q and N are invertible.

Construction of the Explicit Solution of mNLS

- Construct the $p \times p$ matrix valued function $F(x,t)$ as

$$F(x,t) = e^{2A^\dagger x + \frac{1}{2}(A^\dagger)^{-1} t} + Q\, e^{-2Ax - \frac{1}{2} A^{-1} t} N,$$

or, alternatively, the matrix valued function $E(x,t)$ as

$$E(x,t) = e^{2Ax + \frac{1}{2} A^{-1} t} + P\, e^{-2Ax - \frac{1}{2} A^{-1} t} P.$$

- Construct the scalar function $u(x,t)$ via

$$u(x,t) = -4 \int_x^\infty dr\, B^\dagger F(r,t)^{-1} C^\dagger,$$

or

$$u(x,t) = -4 \int_x^\infty dr\, C E(r,t)^{-1} B,$$

Note that $u(x,t)$ is **uniquely constructed** from the triplet (A, B, C) and **exists in every point on the** xt**-plane where the matrix** $F(x,t)$ **(or** $E(x,t)$**) is invertible.**

Admissible class

It is natural to looking for a larger class including triplets such that the properties 1.-3. (or *a.*-*d.*) of the previous slides hold. In fact, for every triplet in this class we can repeat the procedure above illustrated obtaining explicit solutions of the sine-Gordon equation.

The triplet (A, B, C) of size p belongs to the *admissible class* $\mathcal{A}$ if:

- The matrices A, B, and C are all real valued.
- The triplet (A, B, C) corresponds to a minimal realization for $\Omega(y, t)$.
- None of the eigenvalues of A are purely imaginary and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex plane.

Admissible class

For any triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ belonging to the admissible class $\mathcal{A}$ the following properties are satisfied:

I. The Lyapunov equations $\tilde{Q}\tilde{A} + \tilde{A}^\dagger \tilde{Q} = \tilde{C}^\dagger \tilde{C}$, $\tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger$ are uniquely solvable, and their solutions (invertibles and selfadjoints) are given by:

$$\tilde{Q} = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I + i\tilde{A}^\dagger)^{-1} \tilde{C}^\dagger \tilde{C} (\lambda I - i\tilde{A})^{-1},$$
$$\tilde{N} = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I - i\tilde{A})^{-1} \tilde{B}\tilde{B}^\dagger (\lambda I + i\tilde{A}^\dagger)^{-1}.$$

II. The resulting matrix

$$\tilde{F}(x,t) = e^{2\tilde{A}^\dagger x + \frac{1}{2}(\tilde{A}^\dagger)^{-1} t} + \tilde{Q}\, e^{-2\tilde{A}^\dagger x - \frac{1}{2}(\tilde{A}^\dagger)^{-1} t} \tilde{N}$$

is real valuedand invertible on the entire xt-plane, and the function

$$\tilde{u}(x,t) = -4 \int_x^\infty dr \, B^\dagger \tilde{F}(r,t)^{-1} C^\dagger$$

is a analytic solution to the sine-Gordon equation everywhere on the xt-plane.

Equivalent triplets

We say that two triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent if they lead to the same potential $u(x, t)$.

A natural question is the following: *Starting from one triplet in the admissible class, is it possible to get an equivalent triplet such that the matrices A, B, C are real, give a minimal representation for the kernel $\Omega(y, t)$ and all the eigenvalues of A have positive real parts?*

The answer is: YES, however...

$(\tilde{A}, \tilde{B}, \tilde{C})$ is in the admissible class $\xrightarrow{\text{How construct Equivalent Triplets?}}$ (A, B, C) minimal representation where $\mathrm{Re}\lambda_j > 0$

Triplets in canonical form

Without loss of generality we can take the triplet (A, B, C) in the following form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix},$$

where each A_j is a Jordan block, instead B_j and C_j are in the following form:

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := \begin{bmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{bmatrix}$$

Examples

Using Mathematica or Matlab (Symbolic Toolbox) we can rapidly construct exact solutions:

- A diagonal: N-soliton solutions
- A diagonal 2×2: kink-(anti)kink interactions
- $A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$: breather solutions
- A non diagonalizable: multipole solutions (generalizing Olmedilla (1987)).

Multipole solutions

Let us consider the triplet (A, B, C) with

$$A = \begin{bmatrix} a & -1 & 0 \\ 0 & a & -1 \\ 0 & 0 & a \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_3 & c_2 & c_1 \end{bmatrix},$$

where $a > 0$, and c_1, c_2, c_3 are real constants with $c_3 \neq 0$, using the following formula (equivalent to this obtained in the above slides)

$$u(x,t) = 4\tan^{-1}\left(i\,\frac{\det(I + iM(x,t)) - \det(I - iM(x,t))}{\det(I + iM(x,t)) + \det(I - iM(x,t))} \right),$$

where $M(x,t) := e^{-\beta/2} P e^{-\beta/2}$ we obtain

$$\text{num} = c_3^3 e^{-4ax - t/a} + 32g, \qquad \text{den} = 4ae^{-2ax - t/(2a)}[128a^8 e^{4ax + t/a} + h_1 + h_2],$$

$$g := (8a^4 c_1 + 8a^3 c_2 + 8a^2 c_3) - (4a^2 c_2 + 8ac_3)t + c_3 t^2 + (16a^4 c_2 + 16a^3 c_3)x - 8a^2 c_3 xt + 16a^4 c_3 x^2,$$

$$h_1 := (8a^4 c_2^2 - 8a^4 c_1 c_3 + 16a^3 c_2 c_3 + 14a^2 c_3^2) - (4a^2 c_2 c_3 + 4ac_3^2)t,$$

$$h_2 := c_3^2 t^2 + (16a^4 c_2 c_3 + 32a^3 c_3^2)x - 8a^2 c_3^2 tx + 16a^4 c_3^2 x^2.$$

Open Problems

If the time factor $e^{-\frac{1}{2}tA^{-1}}$ is replaced by any nonsingular matrix depending on t and commuting with A, what is the nonlinear evolution equation satisfied by its solutions? EXAMPLES:

modified Korteweg de Vries equation: $\longrightarrow e^{8tA^3}$;
NLS equation: $\longrightarrow e^{4itA^2}$;
???? $\longrightarrow \varphi(t)$ with $\varphi(t)A = A\varphi(t)$

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