Exact Solutions to the modified Korteweg-de Vries equation

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Introduction

Many 1-D nonlinear evolution equations allow explicit solutions that can be obtained by the **I**nverse **S**cattering **T**ransform (IST).

In this talk we consider the modified Korteweg de-Vries (mKdV) equation, i.e.

$$u_t + u_{xxx} + 6|u|^2 u_x = 0\,,$$

where u represents a real scalar function, $(x,\, t) \in \mathbb{R}^2$ and the subscripts denote the appropriate partial derivates.

mKdV equation

This equation arises in many applicative contexts:

- Dynamics of thin elastic rods
- Phonons in anharmonic lattices
- Traffic congestion
- Hyperbolic surfaces

In 1973 Wadati proved that the following initial value problem for the mKdv equation

$$\begin{cases} u_t + u_{xxx} + 6|u|^2 u_x = 0, \\ u(x, 0) \end{cases}$$

is solvable by using the IST method.

In order to get the solutions of this equation, it is also possible to apply other methods such as Hirota method or Bäcklund transformations.

mKdV equation

In this talk applying the IST and the Marchenko method, we get explicit solution to the mKdV equation.
Representing the kernels of the Marchenko equation in a separated form by using a suitable triplet of constant real matrices (A, B, C), we explicitly solve the Marchenko equations by separation of variables. The solution of the mKdV equation is related (by a simple algebraic relation) to the Marchenko equation.

The main result is as follows:
Using a "suitable" triplet of constant real matrices (A, B, C) where A, B, C have dimensions $p \times p$, $p \times 1$ and $1 \times p$, we obtain this solution formula:

$$u(x,t) = -2B^\dagger F^{-1}(x,t)C^\dagger .$$

with $F(x,t) = e^{2A^\dagger x - 8(A^\dagger)^3 t} + Qe^{-2Ax+8A^3 t}N$, and

$$Q = \int_0^\infty ds\, e^{-A^\dagger s} C^\dagger C e^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar} BB^\dagger e^{-A^\dagger r} .$$

Advantages

This procedure has several advantages:

1. It is generalizable to the matrix version of the mKdV and to other (matrix) nonlinear evolution equations
2. The explicit formulas found is expressed in a concise form in terms of the triplet (A, B, C). Using computer algebra, we can "unzip the solution in terms of exponential, trigonometric, and polynomial functions of x and t. Even for matrices A of moderate order, this unzipped expression may take several pages!
3. Our solution formula contains, as special cases, many of the solutions already known in literaure (**one-soliton solution, breaher solutions**). However, our solution formula allow us to construct also a new class of solutions (**multipole solutions**)
4. Choosing different triplets as input in our formula, we get a set of solutions to the mKdV equation which can be used for "validation" of numerical methods.

mKdV and IST

Let us consider the initial value problem for mKdV equation

$$\begin{cases} u_t + u_{xxx} + 6|u|^2 u_x = 0, \\ u(x,0) = q(x) \end{cases}$$

where $u = u(x,t)$ is a real valued function.

In order to solve this problem by using the inverse scattering transform (IST) method, it is necessary to build the direct and inverse scattering for the so-called Zakharov-Shabat system

$$\frac{d\xi}{dx} = -i\lambda\xi + q(x)\,\eta,$$
$$\frac{d\eta}{dx} = i\lambda\eta - q(x)\,\xi,$$

where λ is the spectral parameter, $x \in \mathbb{R}$, and we assume the entries of the potential $q(x)$ is in $L^1(\mathbb{R})$.

Scheme of the IST

The following diagram illustrate the IST scheme:

$$
\begin{array}{ccc}
u(x,0) = q(x) & \xrightarrow{\text{direct scattering at } t=0} & \{R(\lambda,0), \{\lambda_j, c_j\}\} \\
\Big\downarrow \text{mKdV solution} & & \Big\downarrow \text{time evolution} \\
q(x,t) & \xleftarrow[\text{inverse scattering at } t]{} & \{R(\lambda)e^{8\lambda_j^3 t}, \{\lambda_j, c_j\, e^{-8\lambda_j^3 t}\}\}
\end{array}
$$

where $q(x,t)$ satisfy the mKdV equation.

Direct scattering: Jost solutions of the Zakharov-Shabat system

Jost matrix solutions

$$\Psi_r(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_2 + o(1)], & x \to +\infty, \\ e^{-i\lambda Jx} a_l(\lambda) + o(1), & x \to -\infty, \end{cases}$$

$$\Phi_l(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix} = \begin{cases} e^{-i\lambda Jx}[I_2 + o(1)], & x \to -\infty, \\ e^{-i\lambda Jx} a_r(\lambda) + o(1), & x \to +\infty, \end{cases}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $a_l(\lambda)$, $a_r(\lambda)$ are transition matrices.

$$\Psi_r(x,\lambda)a_r(\lambda) = \Phi_l(x,\lambda), \qquad \Phi_l(x,\lambda)a_l(\lambda) = \Psi_r(x,\lambda),$$
$$a_l(\lambda) = a_r(\lambda)^{-1} = a_r(\lambda)^{\dagger}.$$

RED denotes analyticity in $\mathbb{C}^+$ while *BLUE* analyticity in $\mathbb{C}^-$ and the $\dagger$ symbol denotes transpose conjugation.

Riemann-Hilbert problem: the scattering coefficients

$$F_+(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix}, \qquad F_-(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}.$$

As a result, we arrive at the Riemann-Hilbert problems:

$$F_-(\lambda,x) = F_+(\lambda,x)JS(\lambda)J, \quad F_+(\lambda,x) = F_-(\lambda,x)J\overline{S}(\lambda)J, \;\; \lambda \in \mathbb{R}$$

where

$$S(\lambda) = \begin{pmatrix} T_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix}, \; \overline{S}(\lambda) = \begin{pmatrix} \overline{T}_r(\lambda) & \overline{L}(\lambda) \\ \overline{R}(\lambda) & \overline{T}_l(\lambda) \end{pmatrix}$$

are called the *scattering matrix*. The scattering matrices $S(\lambda)$ and $\overline{S}(\lambda)$ are each the inverse of the other and J-unitary, i.e.

$$\overline{S}(\lambda) = S(\lambda)^{-1} = JS^\dagger(\lambda)J, \quad \lambda \in \mathbb{R}$$

.

Inverse scattering of the Zakharov-Shabat system

In order to (re)-construct the potential we have the following procedure:

1. Given the scattering data $\{R(\lambda, t),\, \lambda_j,\, c_{js}(t)\}$, we build the following function

$$\Omega(y;t) = \hat{R}(y;t) + \sum_{j=m+1}^{m+n} \sum_{s=0}^{\eta_j - 1} c_{js}(t) \frac{y^s}{s!}\, e^{i\lambda_j y}.$$

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2. Using the function $\Omega_l(y;t)$, we can consider the following Marchenko integral equation

$$K(x,y;t) - \Omega(y+x;t)^\dagger + \int_x^\infty dv \int_x^\infty dr\, K(x,v;t)\Omega(v+r;t)\Omega(r+y;t)^\dagger = 0.$$

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3. The potential $q(x,t)$ is connected to the above equation by means of the following relationship

$$u(x;t) = -2K(x,x;t).$$

Kernel of the Marchenko equations

It is well known that the kernel of the Marchenko equation has to satisfy the following first order PDE:

$$\Omega_t(y;t) + 8\Omega_{yyy}(y;t) = 0\,.$$

This suggests to take the kernel above introduced as (reflectionless case)

$$\Omega(y;t) = C\, e^{-yA+8A^3 t} B,$$

where A is a $p \times p$ real constant matrix, B, C are, respectively, $p \times 1$ and $1 \times p$ real constant matrices, the blue factor is the so-called time factor.

Let us also assume that the eigenvalues of A have positive real parts and (A, B, C) is a minimal triplet, i.e.,

$$\bigcap_{r=1}^{+\infty} \ker CA^{r-1} = \bigcap_{r=1}^{+\infty} \ker B^\dagger (A^\dagger)^{r-1} = \{0\},$$

Note that the triplet yielding a minimal realization is unique up to a similarity transformation $(A, B, C) \to (EAE^{-1}, EB, CE^{-1})$ for some unique matrix E.

Two different approaches at the Marchenko integral equation

Recalling that $u(x;t)$ is a real valued function, even the kernel $\Omega(y;t)$ and the solution $K(x,y;t)$ of the Marchenko integral equation are real valued:

$$\Omega(y;t)^* = \Omega(y;t),$$
$$K(x,y;t)^* = K(x,y;t).$$

where the $*$ denotes the conjugation. We observe that the kernel $\Omega(y;t)$ (constructed via the triplet (A,B,C)) is a scalar function so that we also have $\Omega(y;t)^\dagger = \Omega(y;t)^*$.

In order to solve the Marchenko equation we can proceed in two different way:

- Approach 1: we can substitute the kernel $\Omega(y;t) = C\, e^{-yA-8A^3t}B$ in the Marchenko equation and consider it as a complex function;
- Approach 2: we take into account that $\Omega(y;t)^\dagger = \Omega(y;t)$ from the beginning when we start to solve the corresponding Marchenko equation.

Explicit expression for the potential: approach 1

Putting

$$\Omega(y;t) = C\, e^{-yA+8A^3t}B, \quad \Omega(y,t)^\dagger = B^\dagger e^{-A^\dagger y+8(A^\dagger)^3t}C^\dagger,$$

into the Marchenko equation we obtain

$$K(x,y;t) - B^\dagger e^{-A^\dagger(x+y)+8(A^\dagger)^3}C^\dagger +$$
$$\int_x^\infty dv \int_x^\infty dr\, K(x,v;t)C\, e^{-(v+r)A+8A^3t}BB^\dagger e^{-A^\dagger(r+y)+8(A^\dagger)^3t}C^\dagger = 0.$$

Defining

$$Q = \int_0^\infty d\gamma\, e^{-\gamma A^\dagger}C^\dagger Ce^{-\gamma A}, \qquad N = \int_0^\infty dy\, e^{-yA}BB^\dagger e^{-yA^\dagger},$$

and looking for a solution in the following form

$$K(x,y;t) = H(x;t)\, e^{-A^\dagger y+8(A^\dagger)^3t}C^\dagger,$$

we arrive, after some easy and straightforward calculations, at the solution

$$K(x,y;t) = B^\dagger F^{-1}(x;t)e^{-A^\dagger(y-x)}C^\dagger.$$

Explicit expression for the potential: approach 1

The matrix $F(x;t)$ which appear in the solution of the Marchenko equation is defined as

$$F(x;t) = e^{-\beta^\dagger} + Q\, e^{\beta} N,$$

where $\beta = -2Ax + 8A^3 t$.
Moreover, the matrix $F(x;t)$ is invertible on the entire xt-plane and its inverse satisfies the following property: $F^{-1}(x;t) \to 0$ exponentially as $x \to \pm\infty$ for each fixed t.

Recalling the relationship between the solution of the Marchenko equation and the potential we get

$$u(x;t) = -2B^\dagger F(x;t)^{-1} C^\dagger = -2C[F(x;t)^\dagger]^{-1} B,$$

where only at the end we have employed the fact that $u(x;t)$ is a real valued function and $\Omega(y;t)^\dagger = \Omega(y;t)$. Using the properties of the function $F(x;t)$ we see that $u(x;t)$ given above is well defined on the entire xt-plane and decays to 0 as $x \to \pm\infty$ for each fixed t.

Explicit expression for the potential: approach 2

Using $\Omega(y,t)^\dagger = \Omega(y,t) = C\, e^{-yA+8A^3 t} B$, for $y > 0$ the Marchenko equation becomes

$$K(x,y;t) - \left(Ce^{-Ax} - \int_x^\infty dz \int_x^\infty ds\, K(x,z;t)\, Ce^{-Az+8A^3 t} e^{-As}\, BCe^{-A^\dagger s} \right) \cdot e^{-Ay+8A^3 t} B = 0\,, \quad y > x\,.$$

Defining

$$P = \int_0^\infty ds\, e^{-sA} BCe^{-sA}$$

and looking for a solution in the following form

$$K(x,y;t) = H_1(x;t)\, Ce^{-Ay+8(A)^3 t} B,$$

we can solve the Marchenko equation in a similar way as seen in Approach 1, obtaining

$$v(x,t) = -2CE(x,t)^{-1} B,$$

where

$$F(x;t)^\dagger = E(x;t) := e^\beta + P\, e^{-\beta} P, \quad \text{with } \beta = 2Ax - 8A^3 t$$

Construction of the explicit solution

In order to obtain explicit solutions for the mKdV equation, we can proceed, in a more direct way, as indicated below. (In blue the steps of the procedure corresponding to APPROACH 1, in red those corresponding to APPROACH 2)

- Consider a **real** triplet (A, B, C) corresponding a minimal realization for $\Omega(y, t)$ (with A is a $p \times p$, B is a $p \times m$ and C is a $n \times p$) and such that all the eigenvalues of A have positive real parts.
- Construct the constant $p \times p$ matrices Q and N that are the unique solutions, respectively, to the Lyapunov equations

$$QA + A^{\dagger}Q = C^{\dagger}C, \ AN + NA^{\dagger} = BB^{\dagger}.$$

  Equivalently, construct the unique solution of the Sylvester equation

$$AP + PA = BC.$$

Lyapunov equation

Assume that the **real** triplet (A, B, C) corresponds to a minimal realization of the kernel Ω and all the eigenvalues of A have positive real parts. Then:

1. The Lyapunov equations

$$QA + A^{\dagger}Q = C^{\dagger}C, \; AN + NA^{\dagger} = BB^{\dagger}.$$

   are uniquely solvable. We call this solutions Q and N being

$$Q = \int_0^{\infty} d\gamma \, e^{-\gamma A^{\dagger}} C^{\dagger} C e^{-\gamma A}, \qquad N = \int_0^{\infty} dy \, e^{-yA} BB^{\dagger} e^{-yA^{\dagger}},$$

   i.e. the same representation that we have found developing the IST!
2. The constant matrices Q and N are selfadjoint; i.e. $Q = Q^{\dagger}$, $N = N^{\dagger}$.
3. The constant matrices Q and N are invertible.
4. The matrix $F(x,t)$ is invertible on the entire xt-plane implying that $u(x,t)$ is the solution of the mKdV equation on the entire xt-plane.

Construction of the explicit solution of mKdV

- Construct the $p \times p$ matrix valued function $F(x;t)$ as

$$F(x;t) = e^{-\beta^\dagger} + Q\, e^{\beta} N,$$

where $\beta = -2Ax + 8A^3 t$ or, alternatively, the matrix valued function $E(x;t)$ as

$$E(x;t) = e^{\beta} + P\, e^{-\beta} P,$$

with $\beta = 2Ax - 8A^3 t$.

- Construct the scalar function $u(x;t)$ via

$$u(x;t) = 2B^\dagger F(x;t)^{-1} C^\dagger$$

or

$$u(x;t) = -2CE(x,t)^{-1} B.$$

Note that $u(x;t)$ is **uniquely constructed** from the triplet (A, B, C) and **exists in every point on the** xt**-plane where the matrix** $F(x;t)$ **(or** $E(x;t)$**) is invertible.**

Admissible class

It is natural to looking for a larger class including triplets such that the properties 1.-4. of the previous slides hold. In fact, for every triplet in this class we can repeat the procedure above illustrated obtaining explicit solutions of the mKdV equation.

The triplet (A, B, C) of size p belongs to the *admissible class* $\mathcal{A}$ if:

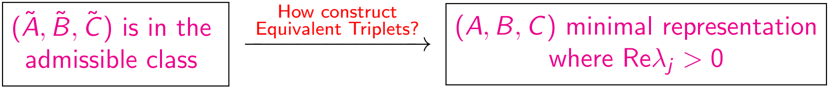
- The matrices A, B, and C are all real valued.
- The triplet (A, B, C) corresponds to a minimal realization for $\Omega(y, t)$.
- None of the eigenvalues of A are purely imaginary and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex plane.

Equivalent triplets

We say that two triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent if they lead to the same potential $u(x, t)$.

A natural question is the following: *Starting from one triplet in the admissible class, is it possible to get an equivalent triplet such that the matrices A, B, C are real, give a minimal representation for the kernel $\Omega(y, t)$ and all the eigenvalues of A have positive real parts?*

The answer is: YES



Reflecting some eigenvalues of A

We have the following:
For any admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, there corresponds an equivalent admissible triplet (A, B, C) in such a way that all eigenvalues of A have positive real parts.

The equivalent triplet (A, B, C) is built as follows:

$$A_1 = \tilde{A}_1, \qquad A_2 = -\tilde{A}_2^{\dagger}, \qquad B_1 = \tilde{B}_1 - \tilde{N}_2 \tilde{N}_4^{-1} \tilde{B}_2, \qquad B_2 = \tilde{N}_4^{-1} \tilde{B}_2,$$

$$C_1 = \tilde{C}_1 - \tilde{C}_2 \tilde{Q}_4^{-1} \tilde{Q}_3, \qquad C_2 = \tilde{C}_2 \tilde{Q}_4^{-1},$$

where A_1 and A_2 contains, respectively, all the eigenvalues having positive real parts and all the eigenvalues having negative real parts.

Reflecting some eigenvalues of A

It is possible to refine the construction in such a way that the triplet (A, B, C) is in a particular **canonical** form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix},$$

where each A_j is a Jordan block, instead B_j and C_j are in the following form:

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := \begin{bmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{bmatrix}$$

Examples: I

Choosing the triplet $(A,\ B,\ C)$ as

$$A = \left(a\right), \quad B = \left(1\right), \quad C = \left(c\right)$$

where $a > 0$ and $c \neq 0 \in \mathbb{R}$, solving the corresponding Sylvester equation, we get

$$P = \left(\frac{c}{2a}\right) .$$

By using the solution formula

$$v(x,t) = -2CE(x,t)^{-1}B,$$

where

$$E(x;t) := e^{\beta} + P\, e^{-\beta} P, \quad \text{with } \beta = 2Ax - 8A^3, t$$

we obtain

$$v(x,\ t) = \frac{-2c}{e^{2ax-8a^3t} + \frac{c^2}{4a^2} e^{-2ax+8a^3t}} ,$$

which may be called a “single-soliton solution.

Examples: II

Let us consider one case in which the transmission coefficients have a pole of order three. This type of situation has never been treated before in literature for the mKdv equation.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1/2 \end{pmatrix}.$$

It is not difficult to verify that the following matrix P satisfies the Sylvester equation

$$P = \begin{pmatrix} 1/8 & 7/16 & 5/8 \\ 1/4 & 3/4 & 13/16 \\ 1/2 & 5/4 & 7/8 \end{pmatrix}.$$

It is not a good idea to unzip the solution formula $v(x,t) = -2CE(x,t)^{-1}B$ in order to write its analytic expression because this representation take a lot of pages!

Examples: II

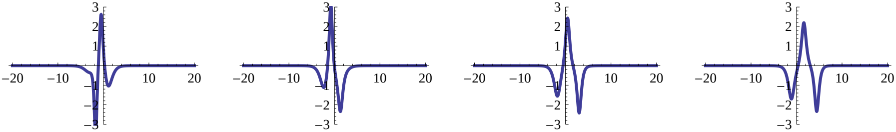


Figura: Four different graphs of the solution corresponding to a triple pole for four fixed values of t ($t = 0$, $t = 1/4$, $t = 1/2$ and $t = 3/4$).

I'm greatly indebted to Antonio Aricò for his assistance in developing the Mathematica code from which this figure was obtained.

Open problems

If the time factor e^{8tA^3} is replaced by any nonsingular matrix depending on t and commuting with A, what is the nonlinear evolution equation satisfied by its solutions? EXAMPLES:

sine-Gordon equation: $\longrightarrow e^{-\frac{1}{2}tA^{-1}}$;
NLS equation: $\longrightarrow e^{4itA^2}$;
???? $\longrightarrow \varphi(t)$ with $\varphi(t)A = A\varphi(t)$

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Acknowledgments

Thank you!!

Scattering Data

Theorem *Assume that the triplet* (A, B, C) *corresponds to a minimal realization for the kernel* $\Omega(y, t)$ *and that all eigenvalues of* A *have positive real parts.*

- $u(x, t) \to 0$ as $x \to +\infty$, tends to an integer multiple of (2π) as $x \to -\infty$.
- For $\lambda \in \mathbf{R}$, the transmission coefficients $T(\lambda)$ and and its inverse are given by

$$T(\lambda) = 1 + iC(\lambda I - iA)^{-1}P^{-1}B.$$
$$T^{-1}(\lambda) = 1 - iCP^{-1}(\lambda I + iA)^{-1}B,$$

  and hence they are functions of λ alone and do not depend on t.
- The reflection coefficient $R(\lambda, t)$ is identically zero.
- The transmission coefficient can be written as the ratio of two determinants as

$$T(\lambda) = \frac{\det(\lambda I + iA)}{\det(\lambda I - iA)}.$$

Admissible class

For any triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ belonging to the admissible class $\mathcal{A}$ the following properties are satisfied:

I. The Lyapunov equations $\tilde{Q}\tilde{A} + \tilde{A}^\dagger\tilde{Q} = \tilde{C}^\dagger\tilde{C}$, $\tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger$ are uniquely solvable, and their solutions (invertibles and selfadjoints) are given by:

$$\tilde{Q} = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I + i\tilde{A}^\dagger)^{-1}\tilde{C}^\dagger\tilde{C}(\lambda I - i\tilde{A})^{-1},$$
$$\tilde{N} = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I - i\tilde{A})^{-1}\tilde{B}\tilde{B}^\dagger(\lambda I + i\tilde{A}^\dagger)^{-1}.$$

II. The resulting matrix

$$\tilde{F}(x,t) = e^{2\tilde{A}^\dagger x + \frac{1}{2}(\tilde{A}^\dagger)^{-1}t} + \tilde{Q}\,e^{-2\tilde{A}^\dagger x - \frac{1}{2}(\tilde{A}^\dagger)^{-1}t}\tilde{N}$$

is real valued and invertible on the entire xt-plane, and the function

$$\tilde{u}(x,t) = -4\int_x^\infty dr\,B^\dagger\tilde{F}(r,t)^{-1}C^\dagger$$

is a analytic solution to the sine-Gordon equation everywhere on the xt-plane.

Sylvester equation

Furtermore, we have also:

a. The Sylvester equation

$$AP + PA = BC.$$

is uniquely solvable. The solution P satisfy

$$P = \int_0^\infty d\gamma\, e^{-\gamma A} BC e^{-\gamma A},$$

i.e. the same representation that we have found via the IST!

b. The constant matrices P is a real function.

c. Because of $NQ = P^2$, P is invertible. Moreover,

$$N(A^\dagger)^j Q = PA^j P, \qquad j = 0, \pm 1, \pm 2, \ldots.$$

which allow us to establish that $E(x;t) = F(x;t)^\dagger$.

d. The matrix $E(x;t)$ is invertible on the entire xt-plane.