SYMMETRIES FOR EXACT SOLUTIONS TO
THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract: A certain symmetry is exploited in expressing exact solutions to the focusing nonlinear Schrödinger equation in terms of a triplet of constant matrices. Consequently, for any number of bound states with any number of multiplicities the corresponding soliton solutions are explicitly written in a compact form in terms of a matrix triplet. Conversely, from such a soliton solution the corresponding transmission coefficients, bound-state poles, bound-state norming constants and Jost solutions for the associated Zakharov-Shabat system are evaluated explicitly. It is also shown that these results hold for the matrix nonlinear Schrödinger equation of any matrix size.

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1. INTRODUCTION

Consider the focusing cubic nonlinear Schrödinger (NLS) equation

\[ iu_t + u_{xx} + 2|u|^2u = 0, \]  

where subscripts denote the appropriate partial derivatives. It arises in applications as diverse as wave propagation in nonlinear media \([15]\), surface waves on sufficiently deep waters \([14,15]\), and signal propagation in optical fibers \([10-12]\). Its initial-value problem is known to be solvable by the inverse scattering transform method \([1,2,13,15]\). In other words, certain solutions to (1.1) can be viewed as a potential in the Zakharov-Shabat system

\[ \frac{d\varphi(\lambda, x, t)}{dx} = \begin{bmatrix} -i\lambda & u(x, t) \\ -u(x, t)^* & i\lambda \end{bmatrix} \varphi(\lambda, x, t), \]  

where an asterisk is used to denote complex conjugation, and \(u(x, t)\) can be recovered from \(u(x, 0)\) with the help of the scattering data sets for (1.2) at \(t = 0\) and at time \(t\).

Exact solutions to nonlinear partial differential equations are of great interest. Such solutions may be helpful to better understand the corresponding nonlinearity, and they may also be useful in producing testing means to determine accuracy of numerical methods for solving nonlinear partial differential equations. This paper is related to exact solutions to (1.1).

In previous papers \([3,6-8]\) we presented a method to construct exact solutions to (1.1) that are globally analytic on the entire \(xt\)-plane and decay exponentially as \(x \to \pm\infty\) at each fixed \(t \in \mathbb{R}\). This has been achieved by using a matrix triplet \((A, B, C)\), where all eigenvalues of \(A\) have positive real parts. A similar method was applied to the Korteweg-de Vries equation on the half-line \([4]\). The same method is also applicable to various other nonlinear partial differential equations that are integrable by the inverse scattering transform with the help of a Marchenko integral equation.

In this paper we analyze the method of \([3]\) when the eigenvalues of the matrix \(A\) in
the triplet \((A, B, C)\) do not all have positive real parts. It is already known that if one or more eigenvalues of \(A\) are purely imaginary, the corresponding scattering coefficients for (1.2) contain discontinuities at some real values of \(\lambda\) and hence the corresponding \(u(x, t)\) cannot be analytic on the entire \(xt\)-plane. Furthermore, it is already known that for soliton solutions to (1.1), the corresponding transmission coefficients for (1.2) must have a pole and a zero appearing as a pair located symmetrically with respect to the real axis, which implies that a pair (or more pairs) of eigenvalues of \(A\) cannot be located symmetrically with respect to the imaginary axis. Thus, in our paper we mainly concentrate on the case where eigenvalues of \(A\) may occur anywhere on the complex plane, but no eigenvalues of \(A\) are on the imaginary axis and no pairs of eigenvalues are located symmetrically with respect to the imaginary axis. For such triplets \((A, B, C)\) we show that there is an equivalent triplet \((\hat{A}, \hat{B}, \hat{C})\) yielding the same solution \(u(x, t)\) to (1.1), where all eigenvalues of \(\hat{A}\) have positive real parts. The corresponding solutions \(u(x, t)\) are then analytic on the entire \(xt\)-plane and they are soliton solutions with any number of poles in the corresponding transmission coefficients and with any multiplicities of such poles. For such triplets we also explicitly evaluate the corresponding transmission coefficients, bound-state norming constants, and the corresponding Jost solutions to (1.2). We also consider the generalization of our results to the matrix case, where the scalar quantity \(u(x, t)\) in the NLS equation is replaced by a matrix-valued function of \(x\) and \(t\).

Our paper is organized as follows. In Section 2 we present the preliminary material by providing an outline of the method of [3] and introduce exact solutions \(u(x, t)\) to (1.1) constructed via a triplet \((A, B, C)\). In Section 3 we exploit a certain symmetry in such exact solutions and show that some (or all) eigenvalues of \(A\) can be chosen either on the right or on the left half complex plane without changing \(u(x, t)\). In Section 4 we show that such solutions are solitons with any number of poles in the corresponding transmission coefficients and with any multiplicities of such poles. We also explicitly evaluate the corresponding transmission coefficients and bound-state norming constants and the Jost
solutions to (1.2). In Section 5 we show that the results of Sections 2-4 obtained for the (scalar) NLS equation (1.1) remain valid for the matrix NLS equation

\[ iu_t + u_{xx} + 2uu^\dagger u = 0, \quad (1.3) \]

where \( u \) is now \( m \times n \) matrix valued, the dagger denotes the matrix adjoint (matrix transpose and complex conjugate), and the associated Zakharov-Shabat system is given by

\[ \frac{d\varphi(\lambda, x, t)}{dx} = \begin{bmatrix} -i\lambda I_m & u(x, t) \\ -u(x, t)^\dagger & i\lambda I_n \end{bmatrix} \varphi(\lambda, x, t), \quad (1.4) \]

with \( I_n \) denoting the \( n \times n \) identity matrix. Finally, we conclude in Section 6 with an explicit example, which was earlier studied as Example 7.2 of [3]. Since one of the eigenvalues of \( A \) in that example has a negative real part, we earlier conjectured that it might be a nonsoliton solution. Using our current results, we verify in Section 6 that it is a two-soliton solution and we explicitly evaluate the corresponding transmission coefficients, the bound-state norming constants, and the Jost solutions to (1.2).

Note that (1.1) and (1.3) are identical when \( u \) is a scalar, and similarly (1.2) and (1.4) are identical when \( u \) is a scalar. Throughout our paper we refer to (1.3) as the scalar NLS equation when \( u \) is a scalar and as the matrix NLS equation when \( u \) is matrix valued. This convenience enables us to state all our results for the NLS equation so that, as shown in Section 5, they remain valid whether the scalar case or the matrix case is considered.

2. PRELIMINARIES

In this section we establish our notation and provide the preliminaries for certain exact solutions to the focusing NLS equation. Such exact solutions are expressed [3] in terms of a triplet of matrices \((A, B, C)\).

Consider any triplet \((A, B, C)\), where \( A \) is a \( p \times p \) (complex-valued) constant matrix, \( B \) is a \( p \times 1 \) (complex-valued) constant matrix, and \( C \) is a \( 1 \times p \) (complex-valued) constant
matrix. For short, we will refer to such a triplet as a triplet of size $p$. From such a triplet, construct the auxiliary $p \times p$ matrices $Q$ and $N$ by solving the respective Lyapunov equations

$$\begin{align*}
QA + A^\dagger Q &= C^\dagger C, \\
AN + NA^\dagger &= BB^\dagger.
\end{align*}$$

(2.1)

Note that if $(Q, N)$ satisfies the system in (2.1), so does $(Q^\dagger, N^\dagger)$; hence, there is no loss of generality in assuming that the solution matrices $Q$ and $N$ to (2.1) are selfadjoint. Then, form the $p \times p$ matrix $F(x, t)$ and the scalar quantity $u(x, t)$ as

$$F(x, t) := e^{2A^\dagger x - 4i(A^\dagger)^2 t} + Q e^{-2Ax - 4iA^2 t} N,$$

(2.2)

$$u(x, t) := -2B^\dagger F(x, t)^{-1} C^\dagger.$$

(2.3)

Let us also define the $p \times p$ matrix $G(x, t)$ and the scalar quantity $v(x, t)$ as

$$G(x, t) := e^{-2Ax - 4iA^2 t} + Ne^{2A^\dagger x - 4i(A^\dagger)^2 t} Q,$$

(2.4)

$$v(x, t) := -2CG(x, t)^{-1} B.$$

(2.5)

**Theorem 2.1** Given a triplet $(A, B, C)$ of size $p$, let $(Q, N)$ be a selfadjoint solution to the system in (2.1). Then $u(x, t)$ defined as in (2.3) satisfies the NLS equation (1.3) at any point on the $xt$-plane where $F(x, t)$ is invertible. Similarly, $v(x, t)$ defined in (2.5) satisfies (1.3) at any point on the $xt$-plane where $G(x, t)$ is invertible.

**PROOF:** Let us only give the proof for $v$ because the proof for $u$ is similar. In fact, $v(x, t)$ turns into $u(-x, t)$ when replacing $(A, B, C)$ with $(A^\dagger, C^\dagger, B^\dagger)$, which can also be used for the proof related to $u(x, t)$. Let us drop the arguments of the functions involved and write (2.4) and its adjoint as

$$G = e^{-\beta} + Ne^{\beta^\dagger} Q, \quad G^\dagger = e^{-\beta^\dagger} + Qe^\beta N,$$

(2.6)

where we have defined

$$\beta := 2Ax + 4iA^2 t.$$

(2.7)
Note that $v$ in (2.5) is well defined as long as $G^{-1}$ exists. Taking appropriate partial derivatives, from (2.5) and (2.6) we obtain

\[ iv_t + v_{xx} + 2vv^\dagger v = -2CG^{-1}PG^{-1}B, \]  

(2.8)

\[ P := -iG_t - G_{xx} + 2G_xG^{-1}G_x + 8BB^\dagger(G^\dagger)^{-1}C^\dagger C. \]

From (2.6) we get

\[ G_t = -4iA^2e^{-\beta} - 4iN(A^\dagger)^2e^{\beta^\dagger}Q, \quad G_x = -2Ae^{-\beta} + 2NA^\dagger e^{\beta^\dagger}Q, \]  

(2.9)

\[ G_{xx} = 4A^2e^{-\beta} + 4N(A^\dagger)^2e^{\beta^\dagger}Q, \]  

(2.10)

\[ e^{\beta^\dagger}QG^{-1} = (G^\dagger)^{-1}Qe^\beta, \quad e^{\beta}N(G^\dagger)^{-1} = G^{-1}Ne^{\beta^\dagger}. \]  

(2.11)

With the help of (2.1), (2.6), and (2.9)-(2.11), one can verify that $P = 0$, and hence the right hand side of (2.8) is zero.

There are several questions that can be raised. For example, are the Lyapunov equations given in (2.1) solvable; if they are solvable, are they uniquely solvable? Are the matrices $F$ and $G$ defined in (2.2) and (2.4), respectively, invertible? The answers to these questions are affirmative under appropriate restrictions on the triplet $(A, B, C)$, as we will see.

Let us note that $u$ and $v$ defined in (2.3) and (2.5), respectively, are analytic functions of $x$ and $t$ at any point on the $xt$-plane as long as the matrices $F$ and $G$, respectively, are invertible at that point. This is because the entries of those matrices and hence also their determinants can be written as sums of products of sine, cosine, exponential, and polynomial functions of linear combinations of $x$ and $t$.

Consider the scalar function $\Omega$ defined as

\[ \Omega(x) := Ce^{-Ax}B. \]  

(2.12)
The right hand side is called a matrix realization of $\Omega$ in terms of the triplet $(A, B, C)$. Without changing $\Omega(x)$, it is possible to increase the value of $p$ in the size of the triplet $(A, B, C)$ by padding the matrices $A$, $B$, $C$ with zeros or by modifying $A$, $B$, $C$ in some other fashion (cf. [8], Subsection 2.4). Conversely, it might also be possible to reduce the value of $p$ in the triplet $(A, B, C)$ so that the quantity $\Omega(x)$ will remain unchanged. The matrix realization in (2.12) is said to be minimal if the value of $p$ in the triplet $(A, B, C)$ is the smallest and yet $\Omega(x)$ remains unchanged by the choice of $p$. The triplet $(A, B, C)$ is minimal if and only if [5] the intersections of the kernels of $CA^j$ and of the kernels of $B^\dagger (A^\dagger)^j$ for $j = 0, 1, 2, \ldots$ are trivial; i.e.

$$\{\xi \in \mathbb{C}^p : CA^j \xi = 0 \text{ for } j \geq 0 \} = \{0\} = \{\eta \in \mathbb{C}^p : B^\dagger (A^\dagger)^j \eta = 0 \text{ for } j \geq 0\}.$$ (2.13)

It is also known [5] that the triplet yielding a minimal realization in (2.12) is unique up to a similarity transformation $(A, B, C) \mapsto (EAE^{-1}, EB, CE^{-1})$ for some unique matrix $E$.

The results in the next theorem are known [3], but they are collected here in a summarized form and a brief proof is included for the benefit of the reader.

**Theorem 2.2** Assume that the triplet $(A, B, C)$ of size $p$ corresponds to a minimal realization in (2.12) and that the eigenvalues of $A$ all have positive real parts. Then:

(i) The Lyapunov equations in (2.1) are uniquely solvable.

(ii) The solutions $Q$ and $N$ are $p \times p$ selfadjoint matrices.

(iii) $Q$ and $N$ can be expressed in terms of the triplet $(A, B, C)$ as

$$Q = \int_0^\infty ds \, [Ce^{-As}]^\dagger [Ce^{-As}], \quad N = \int_0^\infty ds \, [e^{-As}B][e^{-As}B]^\dagger.$$ (2.14)

(iv) $Q$ and $N$ are invertible matrices.

(v) Any square submatrix of $Q$ containing the $(1,1)$-entry or $(p, p)$-entry of $Q$ is invertible. Similarly, any square submatrix of $N$ containing the $(1,1)$-entry or $(p, p)$-entry of $N$ is invertible.
(vi) The quantities $F$ and $G$ defined in (2.2) and (2.4), respectively, are $p \times p$ matrices invertible at every point on the $xt$-plane.

PROOF: By introducing the parameter $\alpha$, let us write the first equation in (2.1) as

$$-Q(\alpha I - A) + (\alpha I + A^\dagger)Q = C^\dagger C,$$

or equivalently as

$$-(\alpha I + A^\dagger)^{-1}Q + Q(\alpha I - A)^{-1} = (\alpha I + A^\dagger)^{-1}C^\dagger C(\alpha I - A)^{-1},$$

(2.15)

where $I$ is the $p \times p$ identity matrix. Since $A$ and $(-A^\dagger)$ have eigenvalues on the right and left complex half planes, respectively, we can integrate (2.15) along a simple and positively oriented contour $\gamma$ lying on the right half complex plane and enclosing all eigenvalues of $A$. Thus, we obtain $Q$ uniquely as

$$Q = \frac{1}{2\pi i} \int_\gamma d\alpha (\alpha I + A^\dagger)^{-1}C^\dagger C(\alpha I - A)^{-1}.$$

Similarly, the solution to the second equation in (2.1) is unique and is obtained as

$$N = \frac{1}{2\pi i} \int_\gamma d\alpha (\alpha I - A)^{-1}BB^\dagger(\alpha I + A^\dagger)^{-1}.$$

Thus, (i) is proved. From (2.1) it is seen that $(Q^\dagger, N^\dagger)$ is a solution to (2.1) whenever $(Q, N)$ is a solution and hence from the uniqueness of the solution we obtain (ii). From (2.14) we see that

$$QA + A^\dagger Q = -\int_0^\infty ds \frac{d}{ds} \left[ e^{-A^\dagger s}C^\dagger Ce^{-As} \right] = -e^{-A^\dagger s}C^\dagger Ce^{-As} \Big|_{s=0}^\infty = C^\dagger C,$$

where we have used the fact that all eigenvalues of $A$ have positive real parts. A similar argument for $N$ completes the proof of (iii). From their selfadjointness and positivity as seen from (2.14), it follows that all eigenvalues of $Q$ and $N$ are nonnegative. Moreover, (2.13) implies that zero is not an eigenvalue of $Q$ and $N$. Hence $Q$ and $N$ are invertible,
proving (iv). The positivity of all eigenvalues also implies (v). The invertibility of $F$ follows from using Theorem 4.2 of [3] in (2.2), and the proof of invertibility for $G$ is similar.

The results in the next theorem are useful in extracting the scattering data for (1.2) from the corresponding potential $u(x,t)$, which is also a solution to (1.1). For the benefit of the reader we state such results in a summarized and unified form. The proofs of these results are available in Theorems 3.1 and 3.3 of [3], and hence they will not be given here.

**Theorem 2.3**

Assume that the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ of size $p$ corresponds to a minimal realization in (2.12) and that the eigenvalues of $\tilde{A}$ all have positive real parts. Further, assume that $\tilde{A}$ has $m$ distinct eigenvalues $\alpha_1, \ldots, \alpha_m$ and the multiplicity of $\alpha_j$ is $n_j$. Then:

(i) There exists a unique triplet $(A, B, C)$, where $A$ is in a Jordan canonical form with each Jordan block containing a distinct eigenvalue and having $-1$ in the superdiagonal entries, and the entries of $B$ consist of zeros and ones. More specifically, we have

\[
A = \begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_m
\end{bmatrix}, \quad
B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_m
\end{bmatrix}, \quad
C = [C_1 \ C_2 \ \ldots \ C_m],
\]

where $A_j$ has size $n_j \times n_j$, $B_j$ has size $n_j \times 1$, $C_j$ has size $1 \times n_j$, and the constants $c_{j(n_j-1)}$ are nonzero.

(ii) The triplet $(A, B, C)$ can be constructed from $(\tilde{A}, \tilde{B}, \tilde{C})$ via

\[
\tilde{A} = MAM^{-1}, \quad \tilde{B} = MSB, \quad C = CMS,
\]

where $M$ is a matrix whose columns are formed from the generalized eigenvectors of $(-\tilde{A})$ and $S$ is an upper triangular Toeplitz matrix commuting with $A$, is uniquely
determined by $M$ and $\tilde{B}$, and has the form

$$S = \begin{bmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_{m+n} \end{bmatrix}, \quad S_j := \begin{bmatrix} \theta_j n_j & \theta_j (n_j - 1) & \ldots & \theta_j 1 \\ 0 & \theta_j n_j & \ldots & \theta_j 2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \theta_j n_j \end{bmatrix},$$

for some constants $\theta_{js}$.

(iii) The triplets $(A, B, C)$ and $(\tilde{A}, \tilde{B}, \tilde{C})$ yield the same solution $u(x, t)$ to (1.1).

(iv) The complex constants $(i\alpha_j)$ correspond to the bound-state poles on the upper half complex plane of the transmission coefficients $T_1$ and $T_r$ appearing in (4.4)-(4.7) of Section 4.

(v) For each $j$, the complex constants $c_{js}$ for $s = 0, 1, \ldots, (n_j - 1)$ appearing in $C$ correspond to the bound-state norming constants associated with the bound-state pole $(i\alpha_j)$ of the transmission coefficients.

3. THE SCALAR NLS EQUATION

Recall that we refer to (1.3) as the scalar NLS equation when $u$ is a scalar quantity and as the matrix NLS equation when $u$ is a matrix quantity. In this section we exploit a certain symmetry in (2.2) and show that without changing the value of the scalar $u(x, t)$ in (2.3) it is possible to transform the triplet $(A, B, C)$ in such a way that some or all eigenvalues of $A$ can be reflected from the right half complex plane to the left half complex plane. The same result holds for the scalar $v(x, t)$ given in (2.5); namely, it remains unchanged when the triplet $(A, B, C)$ is transformed so that some or all eigenvalues of $A$ are reflected with respect to the imaginary axis on the complex plane. Later, we will see that these results remain valid also for the matrix NLS equation when $u$ and $v$ are matrix valued.

For repeated eigenvalues of $A$, the aforementioned transformation must be applied to all the multiplicities in such a way that after the transformation we should not have any
eigenvalue pairs symmetrically located with respect to the imaginary axis of the complex plane. As mentioned in Section 1, eigenvalue pairs symmetrically located with respect to the imaginary axis cannot yield soliton solutions to (1.1). For such pairs, it is already known [9] that (2.1) is not uniquely solvable.

Let us write (2.2) as

\[
F(x, t) = Q e^{-2Ax - 4iA^2 t} + Q^{-1} e^{2A^\dagger x - 4i(A^\dagger)^2 t N^{-1}} N.
\]

Comparing (2.2) and (3.1), we next prove that \( u(x, t) \) appearing in (2.3) remains invariant under the transformation

\[(A, B, C, Q, N) \mapsto (-A^\dagger, -N^{-1}B, -CQ^{-1}, -Q^{-1}, -N^{-1}), \]

where all the eigenvalues of \( A \) are reflected with respect to the imaginary axis on the complex plane as a result of \( A \mapsto (-A^\dagger) \).

**Theorem 3.1** Assume that the triplet \((A, B, C)\) corresponds to a minimal realization in (2.12) and that all eigenvalues of \( A \) have positive real parts. Consider the transformation

\[(A, B, C, Q, N, F, G, u, v) \mapsto (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N}, \tilde{F}, \tilde{G}, \tilde{u}, \tilde{v}), \]

where \((Q, N)\) corresponds to the unique solution to the Lyapunov system in (2.1), the quantities in \((F, G, u, v)\) are as in (2.2)-(2.5),

\[
\tilde{A} = -A^\dagger, \quad \tilde{B} = -N^{-1}B, \quad \tilde{C} = -CQ^{-1}, \quad \tilde{Q} = -Q^{-1}, \quad \tilde{N} = -N^{-1},
\]

and \((\tilde{F}, \tilde{G}, \tilde{u}, \tilde{v})\) is as in (2.2)-(2.5) but by using \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N})\) instead of \((A, B, C, Q, N)\) on the right hand sides. We then have the following:

(a) The quantities \( F \) and \( G \) are transformed as

\[
\tilde{F} = Q^{-1} FN^{-1}, \quad \tilde{G} = N^{-1}GQ^{-1}.
\]
(b) $\tilde{Q}$ and $\tilde{N}$ satisfy the respective Lyapunov equations
\begin{align}
\begin{aligned}
\dot{\tilde{Q}}\tilde{A} + \tilde{A}^\dagger\dot{\tilde{Q}} &= \hat{C}^\dagger\hat{C}, \\
\tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger &= \hat{B}\hat{B}^\dagger.
\end{aligned}
\end{align}
(3.5)

(c) The matrices $\tilde{Q}$ and $\tilde{N}$ are selfadjoint and invertible.

(d) The matrices $\tilde{F}$ and $\tilde{G}$ are invertible at every point on the xt-plane.

(e) $\tilde{u}(x,t) = u(x,t)$ and $\tilde{v}(x,t) = v(x,t)$.

PROOF: Using (3.3) in (2.2) and (2.4) we get (a). Using (2.1) and (3.3), it can directly be verified that (3.5) is satisfied, proving (b). Using (ii) and (iv) of Theorem 2.2 in (3.3), it follows that (c) holds. The invertibility in (d) follows from (3.4) by using the invertibility of $Q$ and $N$ stated in (c) and the invertibility of $F$ and $G$ stated in (vi) of Theorem 2.2. Finally, (e) can be proved by using (2.3) and (2.5) with the help of (3.3), the selfadjointness of $Q$ and $N$, and (3.4).

Next, we show that even if we reflect some of eigenvalues of $A$ from the right to the left half complex plane, Theorem 3.1 remains valid by choosing the transformation in (3.2) appropriately. For this purpose, let us again start with a triplet $(A, B, C)$ of size $p$ and corresponding to a minimal realization in (2.12), where the eigenvalues of $A$ all have positive real parts. Without loss of any generality, let us partition $A, B, C$ as
\begin{align}
A &= \begin{bmatrix} A_1 & 0 \\
0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{3.6}
\end{align}
so that the $q \times q$ block diagonal matrix $A_1$ contains the eigenvalues that will remain unchanged and $A_2$ contains the eigenvalues that will be reflected with respect to the imaginary axis on the complex plane, the submatrices $B_1$ and $C_1$ have sizes $q \times 1$ and $1 \times q$, respectively, and hence $A_2, B_2, C_2$ have sizes $(p-q) \times (p-q), (p-q) \times 1, 1 \times (p-q)$, respectively, for some integer $q$ not exceeding $p$. Let us write the corresponding respective solutions to (2.1) as
\begin{align}
Q &= \begin{bmatrix} Q_1 & Q_2 \\
Q_3 & Q_4 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\
N_3 & N_4 \end{bmatrix}, \tag{3.7}
\end{align}
where $Q_1$ and $N_1$ have sizes $q \times q$, $Q_4$ and $N_4$ have sizes $(p - q) \times (p - q)$, etc. Note that because of the selfadjointness of $Q$ and $N$ stated in Theorem 2.2, we have

$$Q_1^\dagger = Q_1, \quad Q_2^\dagger = Q_3, \quad Q_4^\dagger = Q_4, \quad N_1^\dagger = N_1, \quad N_2^\dagger = N_3, \quad N_4^\dagger = N_4. \quad (3.8)$$

Furthermore, from Theorem 2.2 (v) it follows that $Q_1, Q_4, N_1,$ and $N_4$ are all invertible.

Let us clarify our notational choice in (3.6) and emphasize that the partitioning in (3.6) is not the same partitioning used in (i) of Theorem 2.3.

**Theorem 3.2** Assume that the triplet $(A, B, C)$ partitioned as in (3.6) corresponds to a minimal realization in (2.12) and that all eigenvalues of $A$ have positive real parts. Consider the transformation (3.2) with $(\tilde{A}, \tilde{B}, \tilde{C})$ having similar block representations as in (3.6), $(Q, N)$ as in (3.7) corresponding to the unique solution to the Lyapunov system in (2.2),

$$\tilde{A}_1 = A_1, \quad \tilde{A}_2 = -A_2^\dagger, \quad \tilde{B}_1 = B_1 - N_2 N_4^{-1} B_2, \quad \tilde{B}_2 = -N_4^{-1} B_2, \quad (3.9)$$

$$\tilde{C}_1 = C_1 - C_2 Q_4^{-1} Q_3, \quad \tilde{C}_2 = -C_2 Q_4^{-1}; \quad (3.10)$$

and $(\tilde{Q}, \tilde{N})$ given as

$$\tilde{Q}_1 = Q_1 - Q_2 Q_4^{-1} Q_3, \quad \tilde{Q}_2 = -Q_2 Q_4^{-1}, \quad \tilde{Q}_3 = -Q_4^{-1} Q_3, \quad \tilde{Q}_4 = -Q_4^{-1}, \quad (3.11)$$

$$\tilde{N}_1 = N_1 - N_2 N_4^{-1} N_3, \quad \tilde{N}_2 = -N_2 N_4^{-1}, \quad \tilde{N}_3 = -N_4^{-1} N_3, \quad \tilde{N}_4 = -N_4^{-1}, \quad (3.12)$$

and $(\tilde{F}, \tilde{G}, \tilde{u}, \tilde{v})$ as in (2.2)-(2.5) but by using $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N})$ instead of $(A, B, C, Q, N)$ on the right hand sides. We then have the following:

(a) The quantities $F$ and $G$ are transformed according to

$$\tilde{F} = \begin{bmatrix} I & -Q_2 Q_4^{-1} \\ 0 & -Q_4^{-1} \end{bmatrix} F \begin{bmatrix} I & 0 \\ -N_4^{-1} N_3 & -N_4^{-1} \end{bmatrix}, \quad (3.13)$$

$$\tilde{G} = \begin{bmatrix} I & -N_2 N_4^{-1} \\ 0 & -N_4^{-1} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -Q_4^{-1} Q_3 & -Q_4^{-1} \end{bmatrix}. \quad (3.14)$$
(b) \( \tilde{Q} \) and \( \tilde{N} \) satisfy the respective Lyapunov equations in (3.5).

(c) The matrices \( \tilde{Q} \) and \( \tilde{N} \) are selfadjoint and invertible.

(d) The matrices \( \tilde{F} \) and \( \tilde{G} \) are invertible at every point on the xt-plane.

(e) \( \tilde{u}(x,t) = u(x,t) \) and \( \tilde{v}(x,t) = v(x,t) \).

**Proof:** Let us use \( I \) to denote the identity matrix not necessarily having the same dimension in every appearance in the proof, but that dimension will be apparent to the reader. Note that (3.13) and (3.14) can be verified by using (3.8)-(3.12) in (2.2) and (2.4), which proves (a). The proof of (b) is by direct substitution in (3.5) and by using (2.1) and (3.6)-(3.12) and by noting that

\[
\tilde{B} = \begin{bmatrix}
I & -N_2N_4^{-1} \\
0 & -N_4^{-1}
\end{bmatrix} B = \begin{bmatrix}
I & -N_2 \\
0 & -N_4
\end{bmatrix}^{-1} B,
\]

\[
\tilde{C} = C \begin{bmatrix}
I & 0 \\
-Q_4^{-1}Q_3 & -Q_4^{-1}
\end{bmatrix} = C \begin{bmatrix}
I & 0 \\
-Q_3 & -Q_4
\end{bmatrix}^{-1},
\]

where the invertibility of \( Q_4 \) and \( N_4 \) is also used, which follows from Theorem 2.2 (v). The selfadjointness of \( \tilde{Q} \) and \( \tilde{N} \) follows from (3.7), (3.8), (3.11), and (3.12). The invertibility of \( \tilde{Q} \) and \( \tilde{N} \) can be seen from (3.11) and (3.12) by writing

\[
\tilde{Q} = \begin{bmatrix}
Q_1 & Q_2 \\
0 & I
\end{bmatrix} \begin{bmatrix}
I & 0 \\
-Q_4^{-1}Q_3 & -Q_4^{-1}
\end{bmatrix} = \begin{bmatrix}
Q_1^{-1} & -Q_1^{-1}Q_2 \\
0 & I
\end{bmatrix}^{-1} \begin{bmatrix}
I & 0 \\
-Q_3 & -Q_4
\end{bmatrix},
\]

and a similar expression for \( \tilde{N} \) and the fact that \( Q_1, Q_4, N_1, N_4 \) are all invertible. Thus, (c) is proved. The invertibility in (d) follows from (3.13) and (3.14) by using the invertibility of \( Q_4 \) and \( N_4 \) stated in (v) of Theorem 2.2 and the invertibility of \( F \) and \( G \) stated in (vi) of Theorem 2.2. Finally, we evaluate \( \tilde{F}(x,t) \) by using \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N} \) instead of \( A, B, C, Q, N \) on the right hand side of (2.2) and evaluate \( \tilde{u}(x,t) \) by using \( \tilde{B}, \tilde{F}, \tilde{C} \) instead of \( B, F, C \) on the right hand side of (2.3). With the help of (3.7)-(3.17), it is straightforward to show that \( \tilde{u}(x,t) \) simplifies to \( u(x,t) \); similarly, \( \tilde{v}(x,t) \) simplifies to \( v(x,t) \), completing the proof of (e).
Finally in this section, we show that $u$ and $v$ defined in (2.3) and (2.5), respectively, can be transformed into each other by transforming the triplet $(A, B, C)$ in a particular way.

**Theorem 3.3** Assume that the triplet $(A, B, C)$ corresponds to a minimal realization in (2.12) and that all eigenvalues of $A$ have positive real parts. Consider the transformation (3.2) where $(Q, N)$ corresponds to the unique solution to the Lyapunov system in (2.1) and $(F, G, u, v)$ is as in (2.2)-(2.5),

$$
\tilde{A} = A, \quad \tilde{B} = Q^{-1}C^\dagger, \quad \tilde{C} = B^\dagger N^{-1}, \quad \tilde{Q} = N^{-1}, \quad \tilde{N} = Q^{-1},
$$

and $(\tilde{F}, \tilde{G}, \tilde{u}, \tilde{v})$ is as in (2.2)-(2.5) but by using $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N})$ instead of $(A, B, C, Q, N)$ on the right hand sides. We then have the following:

(a) $\tilde{Q}$ and $\tilde{N}$ satisfy the respective Lyapunov equations given in (3.5).

(b) The matrices $\tilde{Q}$ and $\tilde{N}$ are selfadjoint and invertible.

(c) $\tilde{u}(x, t) = v(x, t)$ and $\tilde{v}(x, t) = u(x, t)$.

**PROOF:** Using (2.1) and (3.18), it can directly be verified that (3.5) is satisfied. The selfadjointness and invertibility of $\tilde{Q}$ and $\tilde{N}$ follow from (3.18) because $Q$ and $N$ have those properties, as stated in Theorem 2.2 (ii) and (iv); hence, (b) holds. Finally, (c) is proved by direct substitution in (2.3) and (2.5) with the help of the selfadjointness of $Q$ and $N$. 

4. THE SCATTERING COEFFICIENTS AND JOST SOLUTIONS

In this section, we evaluate the Jost solutions and the corresponding scattering coefficients for the Zakharov-Shabat system (1.2) associated with the NLS equation (1.3) when the potential is given by (2.3) or (2.5). In analyzing (1.2), we will use the notation $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$, where the subscripts 1 and 2 denote the first and second components. If $\varphi(\lambda, x, t)$ and $\tilde{\varphi}(\lambda, x, t)$ are any two solutions to (1.2), their Wronskian $[\varphi; \tilde{\varphi}]$ is independent
of \( x \), where we have defined

\[
[\varphi; \bar{\varphi}] := \varphi(\lambda^*, x, t) \dagger \bar{\varphi}(\lambda, x, t). \tag{4.1}
\]

We stress that an overbar does not indicate complex conjugation.

Recall [13,15] that the Jost solutions \( \psi, \bar{\psi}, \phi, \) and \( \bar{\phi} \) are defined as those vector solutions to (1.2) with asymptotics

\[
\psi(\lambda, x, t) = \begin{bmatrix} 0 \\
e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad \bar{\psi}(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} \\
0 \end{bmatrix} [1 + o(1)], \quad x \to +\infty, \tag{4.2}
\]

\[
\phi(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} \\
0 \end{bmatrix} [1 + o(1)], \quad \bar{\phi}(\lambda, x, t) = \begin{bmatrix} 0 \\
e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad x \to -\infty. \tag{4.3}
\]

When \( u(\cdot, t) \) is integrable for each fixed \( t \), the four Jost solutions exist and their asymptotics at the other end of the real axis yield the scattering coefficients \( R, L, T_l, T_r \) via

\[
\psi(\lambda, x, t) = \begin{bmatrix} LT_l^{-1} e^{-i\lambda x} \\
T_l^{-1} e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad x \to -\infty, \tag{4.4}
\]

\[
\bar{\psi}(\lambda, x, t) = \begin{bmatrix} (T_l^\dagger)^{-1} e^{-i\lambda x} \\
-L_l^\dagger (T_l^\dagger)^{-1} e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad x \to -\infty, \tag{4.5}
\]

\[
\phi(\lambda, x, t) = \begin{bmatrix} T_r^{-1} e^{-i\lambda x} \\
RT_r^{-1} e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad x \to +\infty, \tag{4.6}
\]

\[
\bar{\phi}(\lambda, x, t) = \begin{bmatrix} -R_l^\dagger (T_l^\dagger)^{-1} e^{-i\lambda x} \\
(T_l^\dagger)^{-1} e^{i\lambda x} \end{bmatrix} [1 + o(1)], \quad x \to +\infty. \tag{4.7}
\]

The scattering coefficients can equivalently be obtained with the help of the Wronskian defined in (4.1); in fact, we have

\[
LT_l^{-1} = (RT_r^{-1})^\dagger = [\varphi; \psi], \quad T_l^{-1} = [\bar{\phi}; \psi], \quad T_r^{-1} = [\bar{\psi}; \phi]. \tag{4.8}
\]

For the Zakharov-Shabat system (1.2) we have \( T_l = T_r \) if the scalar potential \( u(x,t) \) vanishes as \( x \to \pm \infty \). However, we will retain the separate notations for \( T_l \) and \( T_r \) for a subsequent generalization to the matrix case, where \( T_l \) and \( T_r \) may differ.
The Jost solutions $\psi$ and $\bar{\psi}$ and the potential $u(x, t)$ are recovered [13,15] as

$$
\psi(\lambda, x, t) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy \, K(x, y, t) e^{i\lambda y},
$$

(4.9)

$$
\bar{\psi}(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^\infty dy \, \bar{K}(x, y, t) e^{-i\lambda y},
$$

(4.10)

$$
u(x, t) = -2 \begin{bmatrix} 1 & 0 \end{bmatrix} K(x, x, t) = 2 \bar{K}(x, x, t)^\dagger \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

(4.11)

from the solutions to the Marchenko integral equations

$$
\bar{K}(x, y, t) + \begin{bmatrix} 0 \\ \Omega_l(x + y, t) \end{bmatrix} + \int_x^\infty dy \, \bar{K}(x, z, t) \Omega_l(z + y, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y > x,
$$

(4.12)

$$
K(x, y, t) - \begin{bmatrix} \Omega_l(x + y, t)^\dagger \\ 0 \end{bmatrix} - \int_x^\infty dy \, K(x, z, t) \Omega_l(z + y, t)^\dagger = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y > x.
$$

(4.13)

The Jost solutions $\psi$ and $\bar{\psi}$ corresponding to $u(x, t)$ given in (2.3) with $(A, B, C) = (A_l, B_l, C_l)$ can be evaluated by solving (4.12) and (4.13) with

$$
\Omega_l(y, t) = C_l e^{-A_l y - 4iA_l^2 t} B_l.
$$

(4.14)

Since

$$
\Omega_l(x + y, t) = C_l e^{-A_l x} e^{-A_l y - 4iA_l^2 t} B_l,
$$

the Marchenko equations in (4.12) and (4.13) have integral kernels separable in $z$ and $y$ and hence they can be solved explicitly by algebraic methods, yielding

$$
\psi(\lambda, x, t) = \begin{bmatrix} iB_l^\dagger F_l(x, t)^{-1}(\lambda I + iA_l^\dagger)^{-1} e^{i\lambda x} C_l^\dagger \\ e^{i\lambda x} - iC_l [F_l(x, t)^\dagger]^{-1} N_l e^{-2A_l^\dagger x + 4i(A_l^\dagger)^2 t} (\lambda I + iA_l^\dagger)^{-1} e^{i\lambda x} C_l^\dagger \end{bmatrix},
$$

(4.15)

$$
\bar{\psi}(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} + iB_l^\dagger e^{-2A_l^\dagger x + 4i(A_l^\dagger)^2 t} Q_l [F_l(x, t)^\dagger]^{-1}(\lambda I - iA_l)^{-1} e^{-i\lambda x} B_l \\ iC_l [F_l(x, t)^\dagger]^{-1}(\lambda I - iA_l)^{-1} e^{-i\lambda x} B_l \end{bmatrix}.
$$

(4.16)

Similarly, the Jost solutions $\phi$ and $\bar{\phi}$ and the potential $v(x, t)$, in the form given in (2.5), are recovered as

$$
\phi(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^{-\infty} dy \, M(x, y, t) e^{-i\lambda y},
$$

(4.17)
\[ \phi(\lambda, x, t) = \left[ \begin{array}{c} 0 \\ e^{i\lambda x} \end{array} \right] + \int_{-\infty}^{x} dy \tilde{M}(x, y, t) e^{i\lambda y}, \quad (4.18) \]

\[ v(x, t) = 2 \left[ \begin{array}{cc} 1 & 0 \end{array} \right] \tilde{M}(x, x, t) = -2 M(x, x, t)^\dagger \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \quad (4.19) \]

from the solutions to the Marchenko integral equations

\[ \tilde{M}(x, y, t) + \left[ \begin{array}{cc} \Omega_r(x + y, t) & 0 \\ 0 & 0 \end{array} \right] + \int_{-\infty}^{x} dy M(x, z, t) \Omega_r(z + y, t) = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \quad y < x, \quad (4.20) \]

\[ M(x, y, t) - \left[ \begin{array}{c} 0 \\ \Omega_r(x + y, t)^\dagger \end{array} \right] - \int_{-\infty}^{x} dy \tilde{M}(x, z, t) \Omega_r(z + y, t)^\dagger = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \quad y < x. \quad (4.21) \]

Corresponding to the potential \( v(x, t) \) given in (2.5) with \((A, B, C) = (A_r, B_r, C_r)\) we have

\[ \Omega_r(y, t) = C_r e^{A_r y + 4iA_r^2 t} B_r, \quad (4.22) \]

which yields an integral kernel separable in \( z \) and \( y \) in (4.20) and (4.21) and hence allows us to solve those integral equations explicitly by algebraic methods, yielding

\[ \phi(\lambda, x, t) = \left[ \begin{array}{c} e^{-i\lambda x} - iC_r G_r(x, t)^{-1} N_r e^{2A_r^1 x - 4i(A_r^1)^2 t} (\lambda I + iA_r^1)^{-1} e^{-i\lambda x} C_r^1 \\ iB_r^1 [G_r(x, t)^\dagger]^{-1} (\lambda I + iA_r^1)^{-1} e^{-i\lambda x} C_r^1 \end{array} \right], \quad (4.23) \]

\[ \tilde{\phi}(\lambda, x, t) = \left[ \begin{array}{c} iC_r G_r(x, t)^{-1} (\lambda I - iA_r)^{-1} e^{i\lambda x} B_r \\ e^{i\lambda x} + iB_r^1 [G_r(x, t)^\dagger]^{-1} Q_r e^{2A_r x + 4iA_r^2 t} (\lambda I - iA_r)^{-1} e^{i\lambda x} B_r \end{array} \right]. \quad (4.24) \]

**Proposition 4.1** Assume that the triplet \((A, B, C)\) corresponds to a minimal realization in (2.12) and that all eigenvalues of \( A \) have positive real parts. Let \((Q, N)\) correspond to the unique solution to the Lyapunov system (2.1) and let \((F, G)\) be as in (2.2) and (2.4). We then have:

(a) \( F(x, t)^{-1} \to 0 \) and \( G(x, t)^{-1} \to 0 \) as \( x \to \pm \infty \).
(b) \( e^{-2A_1^1 x + 4i(A_1^1)^2 t} Q [F(x, t)^\dagger]^{-1} \to N^{-1} \) as \( x \to -\infty \).
(c) \[ [F(x, t)^\dagger]^{-1} N e^{-2A_1^1 x + 4i(A_1^1)^2 t} \to Q^{-1} \] as \( x \to -\infty \).
(d) \[ [G(x, t)^\dagger]^{-1} Q e^{2A_2 x + 4iA_2^2 t} \to N^{-1} \] as \( x \to +\infty \).
(e) \( G(x,t)^{-1}N e^{2A^t x - 4i(A^1)^2 t} \to Q^{-1} \) as \( x \to +\infty \).

PROOF: The proofs involving \( G \) are obtained by using (2.6) the same way as in the proofs for \( F \), and hence we will only give the proofs for \( F \). Using (2.2) and (2.7) we see that
\[
F = e^{\beta^t} + Qe^{-\beta}, \quad F^\dagger = e^{\beta} + Ne^{-\beta^t}Q,
\]
and hence
\[
F = e^{\beta^t}[I + e^{-\beta^t}Qe^{-\beta}N], \quad Q^{-1}FN^{-1}e^{\beta} = I + Q^{-1}e^{\beta^t}N^{-1}e^{\beta}, \tag{4.25}
\]
\[
F^\dagger Q^{-1}e^{\beta^t}N^{-1} = [I + e^{\beta}Q^{-1}e^{\beta^t}N^{-1}], \quad e^{\beta^t}N^{-1}F^\dagger = [I + e^{\beta^t}N^{-1}e^{\beta}Q^{-1}]Q. \tag{4.26}
\]
Forming the inverses from (4.25) and (4.26) we obtain
\[
F^{-1} = [I + e^{-\beta^t}Qe^{-\beta}N]^{-1}e^{-\beta^t}, \quad F^{-1} = N^{-1}e^{\beta}[I + Q^{-1}e^{\beta^t}N^{-1}e^{\beta}]^{-1}Q^{-1}, \tag{4.27}
\]
\[
e^{-\beta^t}Q(F^\dagger)^{-1} = N^{-1}[I + e^{\beta}Q^{-1}e^{\beta^t}N^{-1}]^{-1}, \tag{4.28}
\]
\[
(F^\dagger)^{-1}Ne^{-\beta^t} = Q^{-1}[I + e^{\beta^t}N^{-1}e^{\beta}Q^{-1}]^{-1}. \tag{4.29}
\]
By letting \( x \to \pm\infty \) in (4.27) we prove (a). By letting \( x \to -\infty \) in (4.28) and (4.29) we prove (b) and (c), respectively. The remaining proofs involving \( G \) are obtained in a similar manner. \( \blacksquare \)

The following result is known [5], but it will be useful later and hence we provide a brief proof.

**Proposition 4.2** For any triplet of matrices \((A, B, C)\) with sizes \( p \times p, p \times n, \) and \( n \times p, \) respectively, we have
\[
[I_n + C(\lambda I_p - A)^{-1}B]^{-1} = I_n - C(\lambda I_p - A + BC)^{-1}B. \tag{4.30}
\]

PROOF: Using the identity
\[
BC = (\lambda I_p - A + BC) - (\lambda I_p - A),
\]
...
one can verify that the product of one side of (4.30) and its inverse yields $I_n$. ■

For convenience we also give a short proof of the following already known result on Schur complements [9].

**Proposition 4.3** For any matrices $U$ and $V$ with sizes $n \times p$ and $p \times n$, respectively, we have the determinant identity

$$\det(I_n + UV) = \det(I_p + VU).$$  \hspace{1cm} (4.31)

**Proof:** Let us use the decompositions

$$\begin{bmatrix} I_p + VU & 0_{pn} \\ 0_{np} & I_n \end{bmatrix} = \begin{bmatrix} I_p & -V \\ 0_{np} & I_n \end{bmatrix} \begin{bmatrix} I_p & V \\ -U & I_n \end{bmatrix} \begin{bmatrix} I_p & 0_{pn} \\ U & I_n \end{bmatrix},$$  \hspace{1cm} (4.32)

$$\begin{bmatrix} I_p & 0_{pn} \\ 0_{np} & I_n + UV \end{bmatrix} = \begin{bmatrix} I_p & 0_{pn} \\ U & I_n \end{bmatrix} \begin{bmatrix} I_p & V \\ -U & I_n \end{bmatrix} \begin{bmatrix} I_p & -V \\ 0_{np} & I_n \end{bmatrix},$$  \hspace{1cm} (4.33)

where $0_{jk}$ denotes the zero matrix of size $j \times k$. Note that the right hand sides in (4.32) and (4.33) have the same determinant, and by equating the determinants of the left hand sides we get (4.31). ■

**Theorem 4.4** Assume that the triplet $(A,B,C)$ corresponds to a minimal realization in (2.12) and that all eigenvalues of $A$ have positive real parts. Let $(Q,N)$ correspond to the unique solution to the Lyapunov system in (2.1), $(F,u)$ be as in (2.2) and (2.3), and the scattering coefficients be defined as in (4.4)-(4.7). We then have:

(a) $u(x,t) \to 0$ as $x \to \pm\infty$.

(b) For $\lambda \in \mathbb{R}$, the transmission coefficients $T_1$ and $T_r$ appearing in (4.4) and (4.6), respectively, and their inverses are given by

$$T_1^{-1} = 1 - iCQ^{-1}(\lambda I + iA^\dagger)^{-1}C^\dagger,$$  \hspace{1cm} (4.34)

$$T_1 = 1 + iC(\lambda I - iA)^{-1}Q^{-1}C^\dagger,$$  \hspace{1cm} (4.35)

20
\[ T_r^{-1} = 1 - iB^\dagger(\lambda I + iA^\dagger)^{-1}N^{-1}B, \quad (4.36) \]
\[ T_r = 1 + iB^\dagger N^{-1}(\lambda I - iA)^{-1}B, \quad (4.37) \]

and hence they are functions of \( \lambda \) alone and do not depend on \( t \).

(c) The reflection coefficients \( L(\lambda, t) \) and \( R(\lambda, t) \) appearing in (4.4) and (4.6) are both identically zero.

(d) The transmission coefficients can be written as the ratio of two determinants as
\[ T_i(\lambda) = \frac{\det(\lambda I + iA^\dagger)}{\det(\lambda I - iA)}, \quad T_i(\lambda) = \frac{\det(\lambda I + iA^\dagger)}{\det(\lambda I - iA)}. \quad (4.38) \]

PROOF: Let us compare (4.4) and (4.5) with (4.15) and (4.16) by ignoring the subscript \( l \) in (4.15) and (4.16). Using Proposition 4.1 (a) in the first component of (4.15), we see that \( LT_l^{-1} = 0 \). Then, with the help of (4.8) we also conclude \( RT_r^{-1} = 0 \). Using Proposition 4.1 (c) in the second component of (4.15) we obtain (4.34), and similarly by using Proposition 4.1 (b) in the first component in (4.16) we obtain
\[ (T_r^{-1})^\dagger = 1 + iB^\dagger N^{-1}(\lambda I - iA)^{-1}B. \quad (4.39) \]

Applying Proposition 4.2 on (4.34) and (4.36) and simplifying the resulting expressions with the help of (2.1), we obtain (4.35) and (4.37). We then get (c) from \( LT_l^{-1} = 0 \) and \( RT_r^{-1} = 0 \). Applying Proposition 4.3 onto (4.35) and (4.37) and using (2.1) we obtain (4.38).

5. THE MATRIX NLS EQUATION

In this section we show that all the results presented in Sections 2-4 for the scalar NLS equation remain valid for the matrix NLS equation as well. In other words, when the scalar function \( u(x, t) \) in (1.3) is replaced with an \( m \times n \) matrix-valued function of \( x \) and \( t \), our results in the previous sections remain valid.
In Section 2 we have started with the triplet \((A, B, C)\), where the sizes of \(A\), \(B\), \(C\) were \(p \times p\), \(p \times 1\), and \(1 \times p\), respectively. In Sections 2-4 we have carefully stated all our results so that they all will remain valid also for the \(m \times n\) matrix NLS equation in (1.3). In order for the reader to see the validity of all the results in Sections 2-4 in the matrix case, in this section we will use subscripts to indicate matrix dimensions by writing \(A\) as \(A_{pp}\), \(B\) as \(B_{pm}\), \(C\) as \(C_{np}\), etc. At appropriate places by interpreting 1 as the identity matrix \(I_m\) or \(I_n\), and interpreting 0 as one of the four zero matrices \(0_{mm}\), \(0_{nn}\), \(0_{mn}\), \(0_{nm}\), we will show that all the results in Sections 2-4 remain valid also in the matrix case.

To produce exact solutions to the \(m \times n\) matrix NLS equation (1.3), we start with a triplet \((A_{pp}, B_{pm}, C_{np})\). The Lyapunov equations in (2.1) remain unchanged with \(Q_{pp}\) and \(N_{pp}\) as the solutions. Theorem 2.1 and its proof remain unchanged, with the matrices \(F_{pp}\) and \(u_{mn}\) still defined as in (2.2) and (2.3), respectively. Note that \(v(x, t)\) defined in (2.5) now becomes an \(n \times m\) matrix, and Theorem 2.1 remains valid with \(v_{nm}\) as in (2.5) and \(G_{pp}\) as given in (2.4) for the \(n \times m\) matrix NLS equation

\[
i \partial \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + 2vv^\dagger v = 0.
\]

Theorem 2.2 and its proof hold verbatim with \(Q_{pp}\), \(N_{pp}\), \(F_{pp}\), \(G_{pp}\) in the matrix case. Alternatively, in defining \(v\) one can simply use a triplet \((A, B, C)\) with sizes \(p \times p\), \(p \times n\), \(m \times p\), respectively, so that \(v\) has the same size as \(u\), in which case Theorem 2.1 and its proof hold verbatim for the \(m \times n\) matrix NLS equation. In fact, this alternative is preferred, because in that case the generalizations of all the results in Sections 2-4 to the matrix case are simpler to state. Theorem 2.3 has a counterpart in the matrix case, but we will not consider that generalization in this paper.

In terms of \(u_{mn}\), \(v_{nm}\), \(\tilde{u}_{nm}\), and \(\tilde{v}_{nm}\), all the results in Section 3 until Theorem 3.3 also hold verbatim in the matrix case. Theorem 3.3 and its proof also hold when stated with \(u_{mn}\), \(v_{nm}\), \(\tilde{u}_{nm}\), and \(\tilde{v}_{mn}\). Alternatively, in evaluating \(v\) and \(\tilde{v}\), by using a triplet \((A, B, C)\) with sizes \(p \times p\), \(p \times n\), \(m \times p\), respectively, all the results in Section 3, including Theorem
3.3 and its proof, hold verbatim in the matrix case.

All the results in Section 4 also hold with obvious interpretations of the stated quantities. The matrix Zakharov-Shabat system corresponding to (1.2) now becomes (1.4). The Jost solutions \( \psi_{(m+n)n}, \phi_{(m+n)m}, \bar{\psi}_{(m+n)m}, \bar{\phi}_{(m+n)n} \) are now matrices. In analogy with (4.2) and (4.3), their asymptotics in the matrix case are given by

\[
\psi(\lambda, x, t) = \begin{bmatrix} 0_{mn} \\ e^{i\lambda x} I_n \end{bmatrix} [1 + o(1)], \quad \bar{\psi}(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} I_m \\ 0_{nm} \end{bmatrix} [1 + o(1)], \quad x \to +\infty,
\]

\[
\phi(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} I_m \\ 0_{nm} \end{bmatrix} [1 + o(1)], \quad \bar{\phi}(\lambda, x, t) = \begin{bmatrix} 0_{mn} \\ e^{i\lambda x} I_n \end{bmatrix} [1 + o(1)], \quad x \to -\infty.
\]

The definition of the Wronskian in (4.1) holds in the matrix case, where it is now a square matrix of size either \( m \times m \) or \( n \times n \), depending on the sizes of matrix solutions. The asymptotic relations (4.4)-(4.7) defining the scattering coefficients hold as stated, where now the scattering coefficients \( R_{nm}, L_{mn}, (T_l)_{nn}, \) and \( (T_r)_{mm} \) are matrices. The Fourier relations (4.9) and (4.10) are also valid in the matrix case and are given by

\[
\psi(\lambda, x, t) = \begin{bmatrix} 0_{mn} \\ e^{i\lambda x} I_n \end{bmatrix} + \int_x^\infty dy K(x, y, t) e^{i\lambda y},
\]

\[
\bar{\psi}(\lambda, x, t) = \begin{bmatrix} e^{-i\lambda x} I_m \\ 0_{nm} \end{bmatrix} + \int_x^\infty dy \bar{K}(x, y, t) e^{-i\lambda y},
\]

where \( K(x, y, t) \) is now an \((m + n) \times n\) matrix and \( \bar{K}(x, y, t) \) is an \((m + n) \times m\) matrix. The Marchenko equations in (4.12) and (4.13) remain unchanged with the understanding that the quantity \( \Omega_l(y, t) \) appearing in the kernel is now an \( n \times m \) matrix and we have

\[
\bar{K}(x, y, t) + \begin{bmatrix} 0_{mm} \\ \Omega_l(x + y, t) \end{bmatrix} + \int_x^\infty dy K(x, z, t) \Omega_l(z + y, t) = \begin{bmatrix} 0_{nn} \\ 0_{mn} \end{bmatrix}, \quad y > x,
\]

\[
K(x, y, t) - \begin{bmatrix} \Omega_l(x + y, t)^\dagger \\ 0_{nn} \end{bmatrix} \quad - \int_x^\infty dy \bar{K}(x, z, t) \Omega_l(z + y, t)^\dagger = \begin{bmatrix} 0_{mn} \\ 0_{nn} \end{bmatrix}, \quad y > x,
\]

and the potential \( u_{mn} \) is recovered as in (4.11) via

\[
u(x, t) = -2 \begin{bmatrix} I_m & 0_{mn} \end{bmatrix} K(x, x, t) = 2 \bar{K}(x, x, t)^\dagger \begin{bmatrix} 0_{mn} \\ I_n \end{bmatrix}.
\]
Similarly, the Fourier representations in (4.17) and (4.18) remain true by interpreting them as
\[
\phi(\lambda, x, t) = \left[ e^{-i\lambda x} I_m \right]_{0n}^{0m} + \int_{-\infty}^{x} dy M(x, y, t) e^{-i\lambda y},
\]
\[
\bar{\phi}(\lambda, x, t) = \left[ 0_{mn} e^{i\lambda x} I_n \right] + \int_{-\infty}^{x} dy \bar{M}(x, y, t) e^{i\lambda y},
\]
where now \(M(x, y, t)\) and \(\bar{M}(x, y, t)\) have sizes \((m + n) \times m\) and \((m + n) \times n\), respectively.

In generalizing (4.22) to the matrix case, let us simply use a triplet \((A_r, B_r, C_r)\) with sizes \(p \times p, p \times n, m \times p\), respectively. With the understanding that \(\Omega_r(x, y, t)\) is now a \(m \times n\) matrix, the Marchenko equations (4.20) and (4.21) hold true and can also be written as
\[
\bar{M}(x, y, t) + \left[ \Omega_r(x + y, t) \right]_{0n}^{0m} + \int_{-\infty}^{x} dy M(x, z, t) \Omega_r(z + y, t) = \left[ 0_{mn} \right]_{0n}^{0m}, \quad y < x,
\]
\[
M(x, y, t) - \left[ \Omega_r(x + y, t)^\dagger \right]_{0m}^{0mm} - \int_{-\infty}^{x} dy \bar{M}(x, z, t) \Omega_r(z + y, t)^\dagger = \left[ 0_{mm} \right]_{0m}^{0mm}, \quad y < x,
\]
and the potential \(v_{mn}\) is recovered as in (4.19) via
\[
v(x, t) = 2 \left[ I_m 0_{mn} \right] \bar{M}(x, x, t) = -2 M(x, x, t)^\dagger \left[ 0_{mn} \right]_{0m}^{0mm}.
\]

When the triplet \((A_l, B_l, C_l)\) with matrix sizes \(p \times p, p \times m, n \times p\), respectively, is used in the Marchenko kernel (4.14), the Jost solutions \(\psi_{(m+n)n}\) and \(\bar{\psi}_{(m+n)m}\) on any half line \(x \in (a, +\infty)\) for any real number \(a\) can be expressed as in (4.15) and (4.16) via
\[
\psi(\lambda, x, t) = \left[ iB_l^\dagger F_l(x, t)^{-1}(\lambda I + iA_l^\dagger)\right]_{0n}^{0m} e^{i\lambda x} C_l^\dagger + \left[ e^{i\lambda x} I_n - iC_l[F_l(x, t)^\dagger]^{-1}N_l e^{-2A_l^\dagger x + 4i(A_l^\dagger)^2 t(\lambda I + iA_l) - 1} e^{i\lambda x} C_l^\dagger \right] ,
\]
\[
\bar{\psi}(\lambda, x, t) = \left[ e^{-i\lambda x} I_m + iB_l^\dagger e^{-2A_l^\dagger x + 4i(A_l^\dagger)^2 t Q_l F_l(x, t)^\dagger} (\lambda I - iA_l)^{-1} e^{-i\lambda x} B_l \right]_{0n}^{0m} iC_l [F_l(x, t)^\dagger]^{-1}(\lambda I - iA_l) e^{-i\lambda x} B_l.
\]

Similarly, when the matrix triplet \((A_r, B_r, C_r)\) of sizes \(p \times p, p \times n, m \times p\), respectively, is used in the Marchenko kernel (4.22), the corresponding Jost solutions \(\phi_{(m+n)m}\) and
\( \tilde{\phi}_{(m+n)n} \) on any half line \( x \in (-\infty, a) \) for any real number \( a \) can be expressed as in (4.23) and (4.24) via

\[
\phi(\lambda, x, t) = \begin{bmatrix}
  e^{-i\lambda x} I_m - i C_r G_r(x, t)^{-1} N_r e^{2A_r^\dagger x - 4i(A_r^\dagger)^2 t} (\lambda I + i A_r^\dagger)^{-1} e^{-i\lambda x} C_r^\dagger \\
  i B_r^\dagger [G_r(x, t)^\dagger]^{-1} (\lambda I + i A_r^\dagger)^{-1} e^{-i\lambda x} C_r^\dagger
\end{bmatrix},
\]

\[
\tilde{\phi}(\lambda, x, t) = \begin{bmatrix}
  e^{i\lambda x} I_n + i B_r^\dagger [G_r(x, t)^\dagger]^{-1} Q_r e^{2A_r^\dagger x + 4i(A_r^\dagger)^2 t} (\lambda I - i A_r)^{-1} e^{i\lambda x} B_r \\
  i C_r G_r(x, t)^{-1} (\lambda I - i A_r)^{-1} e^{i\lambda x} B_r
\end{bmatrix}.
\]

Proposition 4.1 and its proof hold as stated with the understanding that we interpret 0 in (a) as 0\(_{pp}\). Proposition 4.2 already contains the matrix case in (4.30), and Proposition 4.3 is already stated in such a way that it can directly be used in the matrix case. Theorem 4.4 and its proof in the matrix case hold with the following minor differences. The quantities \( u, T_l, T_r, L, R \) are now matrices \( u_{mn}, (T_l)_{nn}, (T_r)_{mm}, L_{mn}, R_{nm} \), respectively; (4.34)-(4.37) and (4.39) hold as stated with the understanding that the scalar 1 becomes an identity matrix \( I \) of appropriate size, namely,

\[
T_l^{-1} = I_n - i C Q^{-1} (\lambda I_p + i A)^{-1} C_r^\dagger,
\]

\[
T_l = I_n + i C (\lambda I_p - i A)^{-1} Q^{-1} C_r^\dagger,
\]

\[
T_r^{-1} = I_m - i B^\dagger (\lambda I_p + i A)^{-1} N^{-1} B,
\]

\[
T_r = I_m + i B^\dagger N^{-1} (\lambda I_p - i A)^{-1} B.
\]

In the matrix case, on the left hand sides of the two equations in (4.38) we need to replace \( T_l \) and \( T_r \) by their determinants; namely, (4.38) in the matrix case becomes

\[
\det T_l(\lambda) = \frac{\det(\lambda I_p + i A^\dagger)}{\det(\lambda I_p - i A)}, \quad \det T_r(\lambda) = \frac{\det(\lambda I_p + i A^\dagger)}{\det(\lambda I_p - i A)}.
\]

6. AN EXAMPLE

We conclude our paper by providing an application of Theorem 3.2 to a specific case. In Example 7.2 of [3] we evaluated the exact solution to the NLS equation corresponding
to the triplet \((A, B, C)\) with
\[
A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad -1].
\] (6.1)

We incorrectly conjectured in that example that we had a nonsoliton solution because one of the eigenvalues, of \(A\) was not positive. Using the result of Theorem 3.2 of the present paper, we are now able to confirm that that solution is indeed a two-soliton solution. For this, we proceed as follows.

Using the triplet \((A, B, C)\) of (6.1), we solve (2.1) in a straightforward manner and get
\[
Q = \begin{bmatrix} 1/4 & -1 \\ -1 & -1/2 \end{bmatrix}, \quad N = \begin{bmatrix} 1/4 & 1 \\ 1 & -1/2 \end{bmatrix}.
\]
We then construct \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N}, \tilde{F})\) via (3.9)-(3.13) and obtain
\[
\tilde{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \tilde{C} = [3 \quad -2],
\] (6.2)
\[
\tilde{Q} = \begin{bmatrix} 9/4 & -2 \\ -2 & 2 \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} 9/4 & 2 \\ 2 & 2 \end{bmatrix},
\]
\[
\tilde{F} = \begin{bmatrix} e^{4x-16it} + \frac{81}{16} e^{-4x-16it} - 4e^{-2x-4it} & \frac{9}{2} e^{-4x-16it} - 4e^{-2x-4it} \\ -\frac{9}{2} e^{-4x-16it} + 4e^{-2x-4it} & e^{2x-4it} - 4e^{-4x-16it} + 4e^{-2x-4it} \end{bmatrix}.
\]

As seen from (6.2), the eigenvalue \((-1)\) of \(A\) is transformed into the eigenvalue \((+1)\) of \(\tilde{A}\).

The potential \(\tilde{u}(x, t)\), or equivalently \(u(x, t)\), is then constructed via (2.3) and we get
\[
u(x, t) = \frac{8e^{4it}(9e^{-4x} + 16e^{4x}) - 32e^{16it}(4e^{-2x} + 9e^{2x})}{-128\cos(12t) + 4e^{-6x} + 16e^{6x} + 81e^{-2x} + 64e^{2x}},
\]
agreeing with the potential of Example 7.2 of [3]. For this potential, the transmission coefficients are evaluated via (4.38) as
\[
T_l(\lambda) = T_r(\lambda) = \frac{(\lambda + 2i)(\lambda + i)}{(\lambda - 2i)(\lambda - i)}.
\]
because the real parts of all eigenvalues of $\tilde{A}$ in the associated triplet ($\tilde{A}, \tilde{B}, \tilde{C}$) are positive. As for the norming constants associated with the bound states at $\lambda = 2i$ and $\lambda = i$, we need to transform the triplet ($\tilde{A}, \tilde{B}, \tilde{C}$) of (6.2) into another triplet $(A, B, C)$, different from (6.1), so that we will have $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. By using (2.16) we obtain

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [9 \ -4].$$

Thus, the norming constant at $\lambda = 2i$ is 9 and the norming constant at $\lambda = i$ is $-4$. Finally, the Jost solutions $\psi(\lambda, x, t)$ and $\tilde{\psi}(\lambda, x, t)$ to the Zakharov-Shabat system (1.2) are obtained via (4.15) and (4.16), respectively, by using $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N}, \tilde{F})$ for $(\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{Q}_l, \tilde{N}_l, \tilde{F}_l)$ there. For example, for the Jost solution $\psi$ we get

$$\psi(\lambda, x, t) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \frac{4ie^{-4x+4it}g_1(\lambda, x, t)}{(\lambda + 2i)(\lambda + i)[-128 \cos(12t) + 4e^{-6x} + 16e^{6x} + 81e^{-2x} + 64e^{2x}]}e^{i\lambda x},$$

where we have defined

$$g_1(\lambda, x, t) := 36(\lambda + i)e^{6x+12it} + 16(\lambda - i)e^{2x+12it} - 16(\lambda + 2i)e^{8x} - 9(\lambda - 2i),$$

$$g_2(\lambda, x, t) := 48(\lambda + i)e^{12it} + 48(\lambda + 2i)e^{-12it} - 6\lambda e^{-6x} - 81(\lambda + i)e^{-2x} - 32(\lambda + 2i)e^{2x}.$$

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