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EXACT SOLUTIONS OF THE MODIFIED KORTEWEG–DE VRIES EQUATION

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We use the inverse scattering method to obtain a formula for certain exact solutions of the modified Korteweg–de Vries (mKdV) equation. Using matrix exponentials, we write the kernel of the relevant Marchenko integral equation as $\Omega(x+y;t) = Ce^{-(x+y)A}e^{8A^3t}B$, where the real matrix triplet (A,B,C) consists of a constant $p\times p$ matrix A with eigenvalues having positive real parts, a constant $p\times 1$ matrix B, and a constant $1\times p$ matrix C for a positive integer p. Using separation of variables, we explicitly solve the Marchenko integral equation, yielding exact solutions of the mKdV equation. These solutions are constructed in terms of the unique solution P of the Sylvester equation $AP+PA=BC$ or in terms of the unique solutions Q and N of the Lyapunov equations $A^\dagger Q+QA=C^\dagger C$ and $AN+NA^\dagger=BB^\dagger$, where $B^\dagger$ denotes the conjugate transposed matrix. We consider two interesting examples.

Keywords: inverse scattering method, Lyapunov equation, explicit solution of the modified Korteweg–de Vries equation

1. Introduction

We consider the focusing modified Korteweg–de Vries (mKdV) equation

$$u_t + u_{xxx} + 6|u|^2u_x = 0, \tag{1.1}$$

where the subscripts denote the corresponding partial derivatives, u denotes a real scalar function, and $(x,\,t)\in\mathbb{R}^2$.

The mKdV equation arises in applications to the dynamics of thin elastic rods [1], phonons in anharmonic lattices [2], meandering ocean currents [3], traffic congestion [4]–[7], hyperbolic surfaces [8], ion acoustic solitons [9], Alfvén waves in collisionless plasmas [10], slag-metallic bath interfaces [11], and Schottky barrier transmission lines [12].

Here, we propose a method for constructing certain exact solutions of (1.1) that are globally analytic on the entire plane (x,t) and decay exponentially as $x\to\pm\infty$ for each fixed $t\in\mathbb{R}$. The method used to obtain these solutions is based on the inverse scattering transform (IST) [13]–[17]. The IST matches (1.1) to the system of first-order ordinary differential equations

$$\frac{d\xi}{dx} = -i\lambda\xi + u(x,t)\eta, \qquad \frac{d\eta}{dx} = i\lambda\eta - u(x,t)\xi, \tag{1.2}$$

known as the Zakharov–Shabat system [18]. We develop its direct and inverse scattering theory. More precisely, writing the corresponding scattering data in terms of a matrix representation [19]–[21], we can separate the variables in the Marchenko integral equation, which allows solving it algebraically. Its solution is easily related to the solution of (1.1).

The method used here has several advantages:

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1. It is easily generalized to the matrix version of Eq. (1.1) and to other (matrix) nonlinear evolution equations (see, e.g., [22]).

2. The explicit formulas found here are expressed concisely in terms of a triplet (A, B, C), where A is a real square matrix of dimension p, C is a real row vector, and B is a real column vector. Using computer algebra, we can "unzip" the solution in terms of exponential, trigonometric, and polynomial functions of x and t. Even for matrices A of moderate order, this unzipped expression can take several pages!

3. Nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants are easily analyzed by our method.

By our method, we can recover known N-soliton and breather mKdV solutions [16], [23], [24].

This paper is organized as follows. In Sec. 2, we develop the direct and the inverse scattering theory for system (1.2) and describe how the IST allows obtaining the solution of (1.1). In Sec. 3, we construct the exact solutions of Cauchy problem (1.1) in terms of real matrices (A, B, C) solving the Marchenko equation. In Sec. 4, we write the triplet (A, B, C) in a "canonical" form. Finally, in Sec. 5, we discuss the one-soliton solution and a multipole solution as examples.

2. The IST method for the mKdV equation

We recall the basic ideas of the IST for a real scalar function u belonging to $L^1(\mathbb{R})$ and introduce the scattering coefficients and Marchenko integral equations. A more detailed exposition can be found in [13]–[17].

We introduce the *Jost functions* from the right $\overline{\psi}(\lambda, x)$ and $\psi(\lambda, x)$ and from the left $\phi(\lambda, x)$ and $\overline{\phi}(\lambda, x)$ and also the right and left Jost matrix solutions $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ as those solutions of (1.2) satisfying the asymptotic conditions

$$\Psi(\lambda, x) = \begin{pmatrix} \overline{\psi}(\lambda, x) & \psi(\lambda, x) \end{pmatrix} = \begin{cases} e^{-i\lambda Jx}\big(I_2 + O(1)\big), & x \to +\infty, \\ e^{-i\lambda Jx} a_\ell(\lambda) + o(1), & x \to -\infty, \end{cases}$$

$$\Phi(\lambda, x) = \begin{pmatrix} \phi(\lambda, x) & \overline{\phi}(\lambda, x) \end{pmatrix} = \begin{cases} e^{-i\lambda Jx}\big(I_2 + o(1)\big), & x \to -\infty, \\ e^{-i\lambda Jx} a_{\mathrm{r}}(\lambda) + o(1), & x \to +\infty, \end{cases}$$

where $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and I_p denotes the $p{\times}p$ identity matrix. We omit the variable t in these formulas. Because this is a first-order system, we obtain

$$\Phi(\lambda, x) = \Psi(\lambda, x) a_{\mathrm{r}}(\lambda), \qquad \Psi(\lambda, x) = \Phi(\lambda, x) a_\ell(\lambda),$$

where $a_{\mathrm{r}}(\lambda)$ and $a_\ell(\lambda)$ are called the *transmission coefficient matrices*. It is easy to prove that for all $\lambda \in \mathbb{R}$, $a_{\mathrm{r}}(\lambda)$ and $a_\ell(\lambda)$ are unitary matrices with a unit determinant and, moreover, they are mutually inverse (see, e.g., [15], [25], [26]). It is convenient to write these matrices in the form

$$a_\ell(\lambda) = \begin{pmatrix} a_{\ell 1}(\lambda) & a_{\ell 2}(\lambda) \\ a_{\ell 3}(\lambda) & a_{\ell 4}(\lambda) \end{pmatrix}, \qquad a_{\mathrm{r}}(\lambda) = \begin{pmatrix} a_{\mathrm{r}1}(\lambda) & a_{\mathrm{r}2}(\lambda) \\ a_{\mathrm{r}3}(\lambda) & a_{\mathrm{r}4}(\lambda) \end{pmatrix},$$

where $a_{\ell k}(\lambda)$ and $a_{\mathrm{r}k}(\lambda)$, $k = 1, 2, 3, 4$, are real scalar functions. For each $x \in \mathbb{R}$, the Jost functions $\phi(\lambda, x)$ and $\psi(\lambda, x)$ [15], [25], [26] are analytic in λ in $\mathbb{C}^+$, continuous in λ in $\overline{\mathbb{C}^+}$, and tend to unity as $|\lambda| \to \infty$

from within $\mathbb{C}^+$, where $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and lower complex open half-planes and $\overline{\mathbb{C}^\pm} = \mathbb{C}^\pm \cup \mathbb{R}$. In contrast, for each $x \in \mathbb{R}$, the Jost functions $\overline{\phi}(\lambda, x)$ and $\overline{\psi}(\lambda, x)$ are analytic in λ in $\mathbb{C}^-$, continuous in λ in $\overline{\mathbb{C}^-}$, and tend to unity as $|\lambda| \to \infty$ from within $\mathbb{C}^-$. It is therefore natural to consider the 2×2 matrices of these functions

$$F_+(\lambda, x) = \begin{pmatrix} \phi(\lambda, x) & \psi(\lambda, x) \end{pmatrix}, \qquad F_-(\lambda, x) = \begin{pmatrix} \overline{\psi}(\lambda, x) & \overline{\phi}(\lambda, x) \end{pmatrix}.$$

As a result, for each $x \in \mathbb{R}$, $F_+(\lambda, x)$ is analytic in λ in $\mathbb{C}^+$ and continuous in λ in $\overline{\mathbb{C}^+}$. In contrast, for each $x \in \mathbb{R}$, $F_-(\lambda, x)$ is analytic in λ in $\mathbb{C}^-$ and continuous in λ in $\overline{\mathbb{C}^-}$. Using this information, we obtain the Riemann–Hilbert problem

$$F_-(\lambda, x) = F_+(\lambda, x) J S(\lambda) J, \tag{2.1}$$

where

$$S(\lambda) = \begin{pmatrix} T_{\mathrm{r}}(\lambda) & L(\lambda) \\ R(\lambda) & T_\ell(\lambda) \end{pmatrix}$$

is called the *scattering matrix.*

The direct problem can be formulated as follows: given the potential $u(x)$, to construct the scattering matrix or, equivalently, determine one reflection coefficient $T_{\mathrm{r}}(\lambda)$ or $T_\ell(\lambda)$, the poles (bound states) λ_j of the transmission coefficient $T_{\mathrm{r}}(\lambda)$ or $T_\ell(\lambda)$ (see Theorem 3.16 in [25]), and the norming constants c_{js} corresponding to those poles. Moreover, under the technical assumption that $a_{\ell 1}(\lambda)$, $a_{\mathrm{r}1}(\lambda)$, $a_{\ell 4}(\lambda)$, and $a_{\mathrm{r}4}(\lambda)$ are invertible for each $\lambda \in \mathbb{R}$, we can relate the transmission coefficients to the coefficients in the scattering matrix $S(\lambda)$. More precisely, we have

$$T_{\mathrm{r}}(\lambda) = a_{\mathrm{r}1}^{-1}(\lambda), \qquad R(\lambda) = -a_{\ell 4}^{-1}(\lambda) a_{\ell 3}(\lambda) = a_{\mathrm{r}3}(\lambda) a_{\mathrm{r}1}^{-1}(\lambda),$$
$$T_\ell(\lambda) = a_{\ell 4}^{-1}(\lambda), \qquad L(\lambda) = -a_{\mathrm{r}1}^{-1}(\lambda) a_{\mathrm{r}2}(\lambda) = a_{\ell 2}(\lambda) a_{\ell 4}^{-1}(\lambda).$$

The inverse scattering problem is to (re)construct the unique potential $u(x)$ when the scattering data are given. Here, following [19], [20], we solve this problem using the Marchenko method (see [17], [18]) as follows:

1. From the scattering data $\{R(\lambda), \{\lambda_j, \{c_{js}\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}\}$, we construct the function

$$\Omega(y) \stackrel{\text{def}}{=} \widehat{R}(y) + \sum_{j=m+1}^{m+n} \sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y}, \tag{2.2}$$

where $\widehat{R}(y) = (1/2\pi) \int_0^\infty d\lambda\, R(\lambda) e^{i\lambda y}$ is the Fourier transform of $R(\lambda)$.

2. We solve the Marchenko integral equation

$$K(x, y) - \Omega^\dagger(x + y) + \int_x^\infty dz \int_x^\infty ds\, K(x, z) \Omega(z + s) \Omega^\dagger(s + y) = 0, \tag{2.3}$$

where $y > x$.

3. We construct the potential $u(x)$ using the formula

$$u(x) = -2K(x, x). \tag{2.4}$$

Having presented the direct and inverse scattering problems corresponding to the linear ordinary differential equations associated with the mKdV equation, we now discuss how the IST allows obtaining the solution of the Cauchy problem for (1.1). Using the initial condition $u(x,0)$ as the potential in system (1.2), we develop the direct scattering theory as shown above and build the scattering matrix. Successively, we let the initial scattering data evolve in time. The transmission coefficient does not change with time; consequently, the bound states also do not change. The reflection coefficient $R(\lambda)$ evolves according to $R(\lambda,t) = e^{8\lambda^3 t}R(\lambda)$. It remains to determine the evolution of the norming constants. Extending our previous results on the nonlinear Schrödinger equation [20], [22], [25], [27] to the mKdV equation, we obtain the time evolution of the norming constants:

$$\big(c_{jn_j-1}(t)\cdots c_{j0}(t)\big) = \big(c_{jn_j-1}(0)\cdots c_{j0}(0)\big)e^{-A_j^3 t}, \tag{2.5}$$

where A_j is the matrix defined by Eq. (4.3). Finally, we obtain the mKdV solution by solving the inverse scattering problem using the scattering data $\big\{R(\lambda,t),\big\{\lambda_j,\{c_{js}(t)\}_{s=0}^{n_j-1}\big\}_{j=m+1}^{m+n}\big\}$.

3. Explicit solutions of the mKdV equation

In this section, we obtain two different but equivalent formulas yielding solutions of the mKdV equation. Using the expressions for the evolved reflection coefficient and the evolved norming constants in (2.2), we obtain

$$\Omega(y;t) = \frac{1}{2\pi}\int_0^\infty d\lambda\, R(\lambda)e^{8\lambda^3 t}e^{i\lambda y} + \sum_{j=m+1}^{m+n}\sum_{s=0}^{n_j-1} c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y}, \tag{3.1}$$

which satisfies the first-order partial differential equation

$$\Omega_t(y;t) + 8\Omega_{yyy}(y;t) = 0. \tag{3.2}$$

Following [19]–[21], we write the kernel $\Omega(y)$ introduced in (2.2) in the form

$$\Omega(y) = Ce^{-Ay}B, \quad y \geq 0, \tag{3.3}$$

where A, B, and C are real matrices of the respective sizes $p\times p$, $p\times 1$, and $1\times p$ for some integer p. Based on Eq. (3.2), we can suppose that $\Omega(y;t)$ should be taken in the form

$$\Omega(y;t) = Ce^{-Ay}e^{8A^3 t}B, \quad y \geq 0. \tag{3.4}$$

For reasons to be clarified later, we have some further requirements for the triplet (A,B,C). More precisely, we require that

1. our triplet (A,B,C) be a *minimal representation* of the kernel $\Omega(y)$, i.e,

$$\bigcap_{r=1}^{\infty}\ker CA^{r-1} = \bigcap_{r=1}^{\infty}\ker B^\dagger(A^\dagger)^{r-1} = \{0\}$$

   (see [28] for the details), and

2. all the eigenvalues of the matrix A have positive real parts.

Following the procedure described in the preceding section, we find explicit solutions of the mKdV equation solving Marchenko integral equation (2.3), where the kernel is given by (3.4) and the unknown function K depends on t. We immediately calculate

$$\Omega^\dagger(y;t) = B^\dagger e^{-A^\dagger y} e^{8(A^\dagger)^3 t} C^\dagger, \quad y \geq 0. \tag{3.5}$$

Using (3.4) and (3.5), we transform (2.3) into

$$K(x,y;t) - \left(B^\dagger e^{-A^\dagger x} - \int_x^\infty dz \int_x^\infty ds\, K(x,z;t) C e^{-Az+8A^3 t} e^{-As} B B^\dagger e^{-A^\dagger s} \right) \times \\ \times e^{-A^\dagger y + 8(A^\dagger)^3 t} C^\dagger = 0, \quad y > x. \tag{3.6}$$

If we now seek a solution of this equation in the form

$$K(x,y;t) = H(x,t) e^{-A^\dagger y + 8(A^\dagger)^3 t} C^\dagger \tag{3.7}$$

and introduce the matrices Q and N as

$$Q = \int_0^\infty ds\, e^{-A^\dagger s} C^\dagger C e^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar} B B^\dagger e^{-A^\dagger r}, \tag{3.8}$$

then after some easy calculations, we obtain

$$H(x,t)\Gamma(x,t) = B^\dagger e^{-A^\dagger x}, \tag{3.9}$$

where

$$\Gamma(x,t) = I_p + e^{-A^\dagger x + 8(A^\dagger)^3 t} Q e^{-2Ax+8A^3 t} N e^{-A^\dagger x}. \tag{3.10}$$

Substituting (3.9) in (3.7), we obtain

$$K(x,y;t) = B^\dagger e^{-A^\dagger x} \Gamma^{-1}(x,t) e^{-A^\dagger y + 8(A^\dagger)^3 t} C^\dagger = B^\dagger F^{-1}(x,t) e^{-A^\dagger (y-x)} C^\dagger.$$

Setting

$$F(x,t) = e^{2A^\dagger x - 8(A^\dagger)^3 t} + Q e^{-2Ax+8A^3 t} N, \tag{3.11}$$

we finally obtain the solution formula

$$u(x,t) = -2 B^\dagger F^{-1}(x,t) C^\dagger. \tag{3.12}$$

The matrices Q and N, the matrix $F(x,t)$, and the scalar function $u(x,t)$ respectively introduced in (3.8), (3.11), and (3.12) satisfy the conditions stated in the following theorem.

Theorem 1. *Let the triplet (A,B,C) be real and be a minimal representation of the kernel $\Omega(y;t)$, and let the eigenvalues of A have positive real parts. Then the following statements hold*:

1. *The matrices Q and N are real, positive definite, and self-adjoint, i.e., $Q^\dagger = Q$ and $N^\dagger = N$.*

2. *The matrices Q and N are simultaneously invertible.*

3. *The matrix $F(x,t)$ is invertible on the entire plane (x,t). Moreover, for each fixed t, $F^{-1}(x,t) \to 0$ as $x \to \pm\infty$.*

4. *The real scalar function $u(x,t)$ satisfies* (1.1) *everywhere on the plane (x,t). Moreover, $u(x,t)$ extends to a function that is analytic on the entire plane (x,t) and decays exponentially for each fixed t as $x \to \pm\infty$.*

Proof. The proof of statements 1–3 and of the analyticity and asymptotic behavior of the solution $u(x,t)$ is identical to the proof of items (ii) and (iii) in Theorem 4.2 in [21] (also taking Theorem 4.3 in that paper into account; we therefore refer the reader to that paper for the details). But we can verify directly that our solution (3.12) satisfies Eq. (1.1). For this, we use a slightly different notation. In particular, we write formula (3.12) as

$$u(x,t) = -2B^\dagger e^{-A^\dagger x}\Gamma^{-1}(x,t)e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger, \tag{3.13}$$

where $\Gamma(x,t) = I_p + Q(x,t)N(x)$ with

$$Q(x,t) = e^{-A^\dagger x}e^{8(A^\dagger)^3 t}Qe^{-Ax}e^{8A^3 t}, \qquad N(x) = e^{-Ax}Ne^{-A^\dagger x}$$

and Q and N are defined in (3.8). It is shown in Sec. 4 that these two matrices are the unique solutions of the so-called Lyapunov equations. We recall the rule that if $A(x)$ is an invertible matrix of functions depending on x such that its derivative with respect to x exists, then

$$\frac{d}{dx}\big(A^{-1}(x)\big) = -A^{-1}(x)\left(\frac{d}{dx}A(x)\right)A^{-1}(x). \tag{3.14}$$

Applying this differentiation rule to the function $\Gamma(x,t)$ and differentiating $u(x,t)$ in (3.13) with respect to t, we easily obtain

$$u_t = -16B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger)^3 - QA^3N]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger$$

(here and hereafter, we omit the dependence of $\Gamma(x,t)$, $Q(x,t)$, and $N(x)$ on x and t). If we again apply (3.14) to the function Γ and take the derivative with respect to x, then after some straightforward calculations and also using (4.1a), we obtain

$$(\Gamma^{-1})_x = \Gamma^{-1}A^\dagger + A^\dagger\Gamma^{-1} - 2\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1}. \tag{3.15}$$

Using this formula, we can directly calculate u_x, obtaining

$$u_x = 4B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger) - QAN]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger. \tag{3.16}$$

Computing the derivative of (3.16) and taking (3.15) into account, we obtain

$$\begin{aligned} u_{xx} = 8B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger)^2 + QA^2N - 2(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)] \times \\ \times \Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger. \end{aligned}$$

By similar calculations, we obtain

$$\begin{aligned} u_{xxx} = 16B^\dagger e^{-A^\dagger x}\Gamma^{-1}\big[(A^\dagger)^3 - QA^3N - 3\big((A^\dagger)^2 + QA^2N\big)\Gamma^{-1}(A^\dagger - QAN) - \\ - 3(A^\dagger - QAN)\Gamma^{-1}\big((A^\dagger)^2 + QA^2N\big) + \\ + 6(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\big]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger. \end{aligned}$$

Finally, using (3.16) and taking into account that u is a real scalar function, we obtain

$$\begin{aligned}2|u|^2u_x = u_x u^\dagger u + u u^\dagger u_x = 16B^\dagger e^{-A^\dagger x}\Gamma^{-1}\big[\big((A^\dagger)^2 + QA^2N\big)\Gamma^{-1}(A^\dagger - QAN) + \\ + (A^\dagger - QAN)\Gamma^{-1}\big((A^\dagger)^2 + QA^2N\big) - \\ - 2(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\Gamma^{-1}(A^\dagger - QAN)\big]\Gamma^{-1}e^{-A^\dagger x}e^{8(A^\dagger)^3 t}C^\dagger,\end{aligned}$$

and it is now very simple to see that $u_{xxx} + 6|u|^2u_x = -u_t$.

We now build a different explicit formula, equivalent to the one expressed by (3.12). To obtain this result, we first note that $\Omega(y;t)$ is a real function because the triplet (A,B,C) is real. As a result, $\Omega^\dagger(y;t) = \Omega(y;t)$. Using this relation, we can write Eq. (2.3) as

$$K(x,y;t) - \left(Ce^{-Ax} - \int_x^\infty dz \int_x^\infty ds\, K(x,z;t)Ce^{-Az+8A^3t}e^{-As}BCe^{-A^\dagger s}\right)e^{-Ay+8A^3t}B = 0, \quad y > x.$$

By very similar computations, we obtain the solution of this equation as

$$K(x,y;t) = CE^{-1}(x,t)e^{-A^\dagger(y-x)}B,$$

where

$$E(x,t) = e^{2Ax-8A^3t} + Pe^{-2Ax+8A^3t}P, \qquad P = \int_0^\infty ds\, e^{-As}BCe^{-As}. \tag{3.17}$$

Finally, using Eq. (2.4), we obtain

$$v(x;t) = -2CE^{-1}(x,t)B. \tag{3.18}$$

The next theorem shows the relations between the matrices Q, N, and P and solution formulas (3.12) and (3.18).

Theorem 2. *Let the triplet* (A,B,C) *be real and be a minimal representation of the kernel* $\Omega(y;t)$*, and let the eigenvalues of the matrix* A *have positive real parts. Then the following statements hold*:

1. *The relation* $NQ = P^2$ *holds.*
2. *The matrix* P *is invertible on the entire plane* (x,t)*.*
3. *We have* $E(x,t) = F^\dagger(x,t)$*, and the matrix* $E(x,t)$ *is consequently invertible on the entire plane* (x,t)*.*
4. *For each fixed* t*,* $E^{-1}(x,t) \to 0$ *as* $x \to \pm\infty$*.*
5. *The real scalar function* $v(x,t)$ *satisfies* (1.1) *everywhere on the plane* (x,t)*. Moreover,* $v(x,t)$ *extends to a function that is analytic on the entire plane* (x,t) *and tends to zero exponentially as* $x \to \pm\infty$ *for each fixed* t*.*
6. *Explicit formulas* (3.12) *and* (3.18) *yield the same exact solution of* (1.1) *everywhere on the entire plane* (x,t)*.*

We skip the proof of this theorem because it can be constructed by reformulating the proofs of Theorems 5.2, 5.4, and 5.5 in [21] (with insignificant modifications). But we remark that, first, by very similar calculations to those developed in the proof of Theorem 1, we can directly verify that formula (3.18) satisfies Eq. (1.1) everywhere on the plane (x,t) and, second, because $u(x,t)$ is real and scalar, it follows from the relation $E(x,t) = F^\dagger(x,t)$ that (3.12) and (3.18) yield the same mKdV solution.

We conclude this section with the following theorem.

Theorem 3. *Let the triplet* (A, B, C) *be real and be a minimal representation of the kernel* $\Omega(y; t)$*, and let the eigenvalues of the matrix* A *have positive real parts. Then the solution of the mKdV equation given in the equivalent forms* (3.12) *and* (3.18) *satisfies*

$$u^2(x,t) = \frac{\partial^2 \log(\det E(x,t))}{\partial x^2} = \frac{\partial^2 \log(\det F(x,t))}{\partial x^2}.$$

The proof of this theorem can be found in [20], [21].

4. Canonical form of the triplet (A, B, C)

In this section, we show how we can choose the triplet (A, B, C) in a "canonical form" without loss of generality. To obtain this representation, we begin by finding the explicit solutions of the mKdV given by (3.12) and (3.18) using an "algorithmic" procedure.

We take a real matrix triplet (A, B, C) realizing a minimal representation of the function $\Omega(y; t)$ and such that the eigenvalues of A have positive real parts and consider the equations

$$A^\dagger Q + QA = C^\dagger C, \qquad AN + NA^\dagger = BB^\dagger, \tag{4.1a}$$

$$AP + PA = BC. \tag{4.1b}$$

Equations (4.1a) are the so-called Lyapunov equations, and (4.1b) is known as a Sylvester equation. These equations were studied in detail in [29], where it was proved that they are uniquely solvable under the restrictions on the triplet (A, B, C) presented above. We have the following theorem.

Theorem 4. *Let the triplet* (A, B, C) *be real and be a minimal representation of the kernel* $\Omega(y; t)$*, and let the eigenvalues of the matrix* A *have positive real parts. Then the unique solutions* Q *and* N *of the Lyapunov equations and the unique solution* P *of the Sylvester equation are such that*

1. *the matrices* Q*,* N*, and* P *are real*; Q *and* N *are self-adjoint*;
1. *the matrices* Q *and* N *can be expressed via* (3.8); *the matrix* P *can be expressed via* (3.17); Q *and* N*, moreover, are simultaneously invertible*; *and the matrix* P *is also invertible.*

A proof of this theorem can be found in [21] and [29].

We suppose that we can solve (4.1a) or (4.1b). As a consequence of Theorem 4, their matrix solutions are respectively those introduced by (3.8) or (3.17) in the preceding section. Therefore, knowing the solutions Q and N or P, we can uniquely construct the respective matrix $F(x, t)$ using Eq. (3.11) or $E(x, t)$ using (3.17). Finally, we can write the solution of the mKdV equation by (3.12) or (3.18).

It is natural to seek a larger class including triplets such that the solutions of the corresponding Lyapunov or Sylvester equations have the properties listed in Theorem 4. In fact, for each triplet in this class, we can repeat the procedure introduced above. For this, we introduce the following definitions.

Definition 1. We say that a triplet (A, B, C) of size p belongs to the *admissible class* if the following conditions are satisfied:

a. The matrices A, B, and C are real.

b. The triplet (A, B, C) corresponds to the minimal realization when used in the right-hand side of (3.3).

c. None of the eigenvalues of A is purely imaginary, and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex-λ plane.

Definition 2. Two triplets $(\widetilde{A},\widetilde{B},\widetilde{C})$ and (A,B,C) are said to be *equivalent* if they lead to the same potential $u(x,t)$.

We now consider a triplet $(\widetilde{A},\widetilde{B},\widetilde{C})$ in the admissible class. For such a triplet, we discuss the solutions of the corresponding Lyapunov and Sylvester equations

$$\widetilde{A}^\dagger\widetilde{Q}+\widetilde{Q}\widetilde{A}=\widetilde{C}^\dagger\widetilde{C},\qquad \widetilde{A}\widetilde{N}+\widetilde{N}\widetilde{A}^\dagger=\widetilde{B}\widetilde{B}^\dagger, \tag{4.2a}$$

$$\widetilde{A}\widetilde{P}+\widetilde{P}\widetilde{A}=\widetilde{B}\widetilde{C}. \tag{4.2b}$$

The following theorem was proved in [21], [29].

Theorem 5. *If a triplet $(\widetilde{A},\widetilde{B},\widetilde{C})$ belongs to the admissible class, then the following statements hold*:

1. *Equations* (4.2a) *and* (4.2b) *have unique solutions.*
2. *The matrix solutions $\widetilde{Q}$ and $\widetilde{N}$ of* (4.2a) *are self-adjoint. Moreover, $\widetilde{Q}$ and $\widetilde{N}$ are simultaneously invertible, and $\widetilde{P}$ is also invertible.*
3. *The matrices*

$$\widetilde{F}(x,t)=e^{2\widetilde{A}^\dagger x-8(\widetilde{A}^\dagger)^3t}+\widetilde{Q}e^{-2\widetilde{A}x+8\widetilde{A}^3t}\widetilde{N},\qquad \widetilde{E}(x,t)=e^{2\widetilde{A}x-8\widetilde{A}^3t}+\widetilde{P}e^{-2\widetilde{A}x+8\widetilde{A}^3t}\widetilde{P}$$

   are invertible on the entire plane (x,t).
4. *The functions*

$$\widetilde{u}(x,t)=-2\widetilde{B}^\dagger\widetilde{F}^{-1}(x,t)\widetilde{C}^\dagger,\qquad \widetilde{v}(x,t)=-2\widetilde{C}\widetilde{E}^{-1}(x,t)\widetilde{B}$$

   yield the same explicit solutions of (1.1)*. Moreover, they extend to functions that are analytic on the entire plane (x,t) and tend to zero exponentially as $x\to\pm\infty$ at each fixed t.*

We can then apply the above algorithmic procedure to every triplet in the admissible class.

A natural question now is whether we can start from a triplet $(\widetilde{A},\widetilde{B},\widetilde{C})$ in the admissible class and construct an equivalent triplet (A,B,C) such that the matrices A, B, and C are real and are a minimal representation of the function $\Omega(y;t)$ and the eigenvalues of A have positive real parts (i.e., a triplet (A,B,C) of the type considered in Sec. 3). The answer to this question is affirmative. To save space here, we refer the reader to [20] or [21], where a complete solution of this problem was presented. In particular, Eqs. (4.7) and (4.8) in [21] yield the triplet (A,B,C) by starting from the triplet $(\widetilde{A},\widetilde{B},\widetilde{C})$, while Eqs. (4.10) and (4.11) in that paper explain how to construct Q, N, E, and F from $\widetilde{Q}$, $\widetilde{N}$, $\widetilde{E}$, and $\widetilde{F}$. Consequently, we obtain the following theorem, which establishes the "canonical choice" of the triplet (A,B,C) in (3.4) and hence in explicit formulas (3.12) and (3.18).

Theorem 6. *With any admissible triplet $(\widetilde{A},\widetilde{B},\widetilde{C})$, we can associate a special admissible triplet (A,B,C), where A has the Jordan canonical form with each Jordan block containing a distinct eigenvalue with a positive real part, the column B consists of zeros and ones, and C has real entries. More specifically, for some appropriate positive integer m, we have*

$$A=\begin{pmatrix}A_1&0&\cdots&0\\0&A_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&A_m\end{pmatrix},\qquad B=\begin{pmatrix}B_1\\B_2\\\vdots\\B_m\end{pmatrix},\qquad C=\begin{pmatrix}C_1&C_2&\cdots&C_m\end{pmatrix},$$

where in the case of a real (positive) eigenvalue ω_j of A_j, the corresponding blocks A_j, B_j, and C_j are given by

$$A_j := \begin{pmatrix} \omega_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & \omega_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.3}$$

$$C_j := \begin{pmatrix} c_{jn_j} & c_{jn_{j-1}} & c_{jn_{j-2}} & \cdots & c_{j2} & c_{j1} \end{pmatrix},$$

where A_j has the size $n_j \times n_j$, B_j has the size $n_j \times 1$, C_j has the size $1 \times n_j$, and the constant c_{jn_j} is nonzero. In the case of complex eigenvalues, which must appear in pairs as $\alpha_j \pm i\beta_j$ with $\alpha_j > 0$, the corresponding blocks are given by

$$A_j := \begin{pmatrix} \Lambda_j & -I_2 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \cdots & 0 & 0 \\ 0 & 0 & \Lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \cdots & 0 & \Lambda_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$C_j := \begin{pmatrix} \gamma_{jn_j} & \epsilon_{jn_j} & \gamma_{jn_{j-1}} & \epsilon_{jn_{j-1}} & \gamma_{jn_{j-2}} & \epsilon_{jn_{j-2}} & \cdots & \gamma_{j2} & \epsilon_{j2} & \gamma_{j1} & \epsilon_{j1} \end{pmatrix},$$

where γ_{js} and ϵ_{js} for $s = 1, \ldots, n_j$ are real constants with $\gamma_{jn_j}^2 + \epsilon_{jn_j}^2 > 0$, each column vector B_j has $2n_j$ components, each A_j has the size $2n_j \times 2n_j$, and the 2×2 matrix Λ_j is defined as $\Lambda_j := \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$.

Proof. The real triplet (A, B, C) can be chosen as in Sec. 3 in [20].

5. Significant examples

Example 1. Let $m = 1$. We choose the triplet (A, B, C) in the form

$$A = a, \qquad B = 1, \qquad C = c,$$

where $a > 0$ and $0 \neq c \in \mathbb{R}$ and solve Sylvester equation (4.1b). Finally, we obtain

$$P = \frac{c}{2a}.$$

Using Eq. (3.18), we obtain

$$v(x,t) = -\frac{2c}{e^{2ax-8a^3t} + (c^2/4a^2)\, e^{-2ax+8a^3t}},$$

which can be called a "single-soliton solution" of (1.1) (cf. [16], [23], [24]).

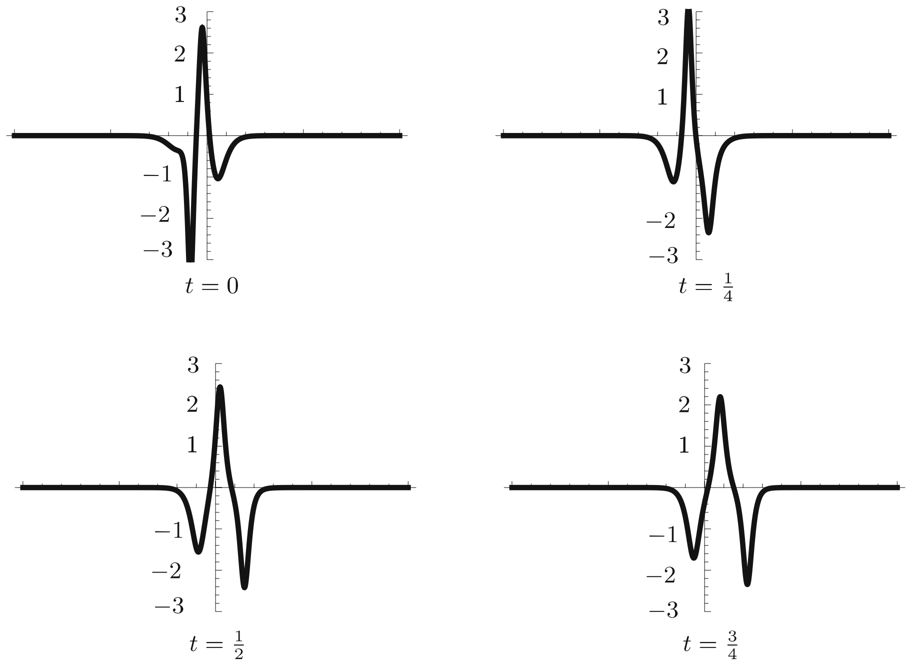

Fig. 1

Example 2. In this example, we consider the case in which the transmission coefficients have a pole of order three. More precisely, we consider the triplet

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 1/2 \end{pmatrix}.$$

It is easy to verify that the matrix

$$P = \begin{pmatrix} 1/8 & 7/16 & 5/8 \\ 1/4 & 3/4 & 13/16 \\ 1/2 & 5/4 & 7/8 \end{pmatrix}.$$

satisfies (4.1b). In this case, it is not a good idea to unzip solution formula (3.18) to write its analytic expression: this representation takes many pages! But it is very easy to plot this solution using the program package Mathematica. Four different graphs of $u(x,t)$ for four fixed values of t ($t = 0$, $t = 1/4$, $t = 1/2$, and $t = 3/4$) are given in Fig. 1.

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