SOME FURTHER CONSIDERATIONS ON THE GALILEAN RELATIVITY PRINCIPLE IN EXTENDED THERMODYNAMICS

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Aim of this paper is to furnish further arguments on the naturalness of the work "A new method to exploit the Entropy Principle and Galilean invariance in the macroscopic approach of Extended Thermodynamics" by Pennisi and Ruggeri; in particular, it will be shown how it was potentially included in a previous work on Galileanity, by Ruggeri. Therefore, the salient points of this work will be here revised, taking care to show the above mentioned application. The notation will be useful also for the paper "The Galilean Relativity Principle as non-relativistic limit of Einstein's one in Extended Thermodynamics" by Carrisi, Pennisi and Scanu, where the same results will be obtained starting from the Einstein' s relativity principle.

1. Expositive part

Recently, (see paper [1]), Ruggeri and Pennisi have found a way to overcome the difficulties arising from the galilean relativity principle in extended thermodynamics. Here we will present new considerations on this subject, which further show how natural the results obtained in [1] are. In particular we will see that it was potentially included in [2]. In fact, it is easy to recognize, in the subsequent eqs. (15), (16), (11) and (13), their counterparts (28) and (29) of [1]. Consequently, the approach there indicated is not simply a mathematical tool to obtain the results, but is what expected from the Galilean relativity principle, except for what concerns the separation of the variables into convective and non convective parts. These results will be obtained also in [3] starting from the Einstein' s relativity principle, thus furnishing further arguments which support them. Some details are not explained because they are familiar to whomever knows the book [4]. Let $F^{i_1\cdots i_n}$ and $F'^{i_1\cdots i_n}$ be the independent variables in two reference frames which are equivalent from the galilean point of view, and let v^i be

the velocity of the points of the second frame, with respect to the first one. The law expressing the change of independent variables is

$$F^{i_1\cdots i_n} = \sum_{h=0}^{n} \binom{n}{h} F'^{(i_1\cdots i_h} v^{i_{h+1}} \cdots v^{i_n)} \qquad \text{for} \quad n = 0, \cdots, N. \tag{1}$$

The same law, for $n = N+1$ gives the transformation of the dependent variable $F^{i_1\cdots i_{N+1}}$. In the sequel, we will use the relation

$$F^{i_1\cdots i_n k} = \sum_{h=0}^{n} \binom{n}{h} F'^{k(i_1\cdots i_h} v^{i_{h+1}} \cdots v^{i_n)} + F^{i_1\cdots i_n} v^k, \tag{2}$$

which can be easily proven.

Now, if h is a scalar function, the galilean relativity principle implies that

$$h(F^{i_1\cdots i_n}) = h(F'^{i_1\cdots i_n}). \tag{3}$$

Before imposing the other conditions, it is better to first change the variables; more precisely, in the literature one usually takes as independent variables the Lagrange multipliers defined by the equations

$$\lambda_{i_1\cdots i_n} = \frac{\partial h}{\partial F^{i_1\cdots i_n}}, \tag{4}$$

which is also the law defining the change of variables. The new ones are also called the "mean field". Let us now see how they transform, under the above change of reference frame. By using the derivation rule of composite functions and eq. (1), we obtain

$$\lambda'_{j_1\cdots j_h} = \sum_{n=0}^{N} \frac{\partial h}{\partial F^{i_1\cdots i_n}} \cdot \frac{\partial F^{i_1\cdots i_n}}{\partial F'^{j_1\cdots j_h}} = \sum_{n=h}^{N} \lambda_{i_1\cdots i_n} \binom{n}{h} \delta^{(i_1}_{j_1} \cdots \delta^{i_h}_{j_h} v^{i_{h+1}} \cdots v^{i_n)},$$

that is

$$\lambda'_{j_1\cdots j_h} = \sum_{n=h}^{N} \binom{n}{h} \lambda_{j_1\cdots j_h j_{h+1}\cdots j_n} v^{j_{h+1}} \cdots v^{j_n}. \tag{5}$$

We note that, until now, we have not imposed the entropy principle nor that h is its density! We have only imposed that if h is a scalar function of the variables $F^{i_1\cdots i_n}$, then it must satisfy the equation (3).

Now, if we assume also that h is the entropy density and ϕ^k is its flux, we know that this last one transforms, under the above change of reference frame, under the law

$$\phi^k = \phi'^k + v^k h. \tag{6}$$

Similarly, in the literature the functions $\tilde{h}$ and $\tilde{\phi}^k$ are defined by

$$\tilde{h} = -h + \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} F^{i_1\cdots i_n}, \quad \tilde{\phi}^k = -\phi^k + \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} F^{i_1\cdots i_n k}, \tag{7}$$

which is a change from the dependent variables h and ϕ^k to $\tilde{h}$ and $\tilde{\phi}^k$. Also for these functions, let us now see how they transform, under the above change of reference frame. To this end, we will exploit the following relations

$$\sum_{m=0}^{N}\lambda_{i_1\cdots i_m}F^{i_1\cdots i_m}=\sum_{n=0}^{N}\lambda'_{i_1\cdots i_n}F'^{i_1\cdots i_n}\,,$$

$$\sum_{m=0}^{N}\lambda_{i_1\cdots i_m}\left(F^{i_1\cdots i_m k}-F^{i_1\cdots i_m}v^k\right)=\sum_{n=0}^{N}\lambda'_{i_1\cdots i_n}F'^{i_1\cdots i_n k}\,. \qquad (8)$$

Let us prove the first of these

$$\sum_{n=0}^{N}\lambda'_{i_1\cdots i_n}F'^{i_1\cdots i_n}=\sum_{n=0}^{N}\sum_{m=n}^{N}\binom{m}{n}\lambda_{i_1\cdots i_n i_{n+1}\cdots i_m}v^{i_{n+1}}\cdots v^{i_m}F'^{i_1\cdots i_n}=$$

$$=\sum_{m=0}^{N}\sum_{n=0}^{m}\binom{m}{n}\lambda_{i_1\cdots i_n i_{n+1}\cdots i_m}v^{(i_{n+1}}\cdots v^{i_m}F'^{i_1\cdots i_n)}=\sum_{m=0}^{N}\lambda_{i_1\cdots i_m}F^{i_1\cdots i_m}\,,$$

where in the second passage eq. (5) has been used, in the third one the order of the 2 summations has been exchanged and symmetrization introduced, and in the fourth one eq. (1) has been used.

The proof of eq. $(8)_2$ is similar, except that in the last passage eq. (2) has to be used.

Now, by using $(7)_1$, eq. $(8)_1$ yields $\tilde{h}'+h'=\tilde{h}+h$ which, by using $h'=h$ yields $\tilde{h}'=\tilde{h}$. The result is that $\tilde{h}$ behaves like a scalar, i.e., has the same value in the 2 reference frames!

Similarly, by using (7), eq. $(8)_2$ yields $\tilde{\phi}^k+\phi^k-(\tilde{h}+h)v^k=\tilde{\phi}'^k+\phi'^k$, which, by using (6) yields $\tilde{\phi}^k-\tilde{h}v^k=\tilde{\phi}'^k$. (9)

In this way we have seen how $\tilde{h}$ and $\tilde{\phi}^k$ transform by changing the reference frame. Now we can impose the galilean relativity principle; because $\tilde{h}$ is a scalar function, we must have $\tilde{h}(\lambda'_{j_1\cdots j_h})=\tilde{h}(\lambda_{j_1\cdots j_h})$ that is

$$\tilde{h}\left(\sum_{n=h}^{N}\binom{n}{h}\lambda_{j_1\cdots j_h j_{h+1}\cdots j_n}v^{j_{h+1}}\cdots v^{j_n}\right)=\tilde{h}(\lambda_{j_1\cdots j_h})\,. \qquad (10)$$

This already holds in $v^j=0$, so that it is equivalent to its derivative with respect to v^j, that is

$$0=\frac{\partial\tilde{h}}{\partial\lambda'_{j_1\cdots j_h}}\sum_{n=h+1}^{N}\binom{n}{h}(n-h)\lambda_{j_1\cdots j_h j j_{h+2}\cdots j_n}v^{j_{h+2}}\cdots v^{j_n}=$$

$$=(h+1)\frac{\partial\tilde{h}}{\partial\lambda'_{j_1\cdots j_h}}\lambda'_{j_1\cdots j_h j}\quad.$$

In other words, $\tilde{h}$ is a scalar function if it satisfies the condition

$$\sum_{h=0}^{N-1}(h+1)\frac{\partial\tilde{h}}{\partial\lambda'_{j_1\cdots j_h}}\lambda'_{j_1\cdots j_h j}=0\,. \qquad (11)$$

Regarding $\tilde{\phi}^k$, the galilean relativity principle implies that the following diagram is commutative.

$$\begin{tikzcd}
\boxed{\lambda_A} \arrow[rr] \arrow[dd, "\tilde{\phi}^k"] & & \boxed{\lambda'_B(\lambda_A)} \arrow[d, "\tilde{\phi}'^k"] \\
& & \boxed{\tilde{\phi}'^k[\lambda'_B(\lambda_A)]} \arrow[d, equal] \\
\boxed{\tilde{\phi}^k(\lambda_A)} \arrow[rr, "\text{(Obtaining } \tilde{\phi}'^k \text{ from eq.(9))}"'] & & \boxed{\tilde{\phi}^k(\lambda_A)-v^k\tilde{h}(\lambda_A)}
\end{tikzcd}$$

In other words

$$\tilde{\phi}'^k\left(\sum_{n=h}^{N}\binom{n}{h}\lambda_{j_1\cdots j_h j_{h+1}\cdots j_n}v^{j_{h+1}}\cdots v^{j_n}\right)=\tilde{\phi}^k(\lambda_{j_1\cdots j_h})-v^k\tilde{h}(\lambda_{j_1\cdots j_h}). \qquad (12)$$

Proceeding as for $\tilde{h}$, we find $\displaystyle\sum_{h=0}^{N-1}(h+1)\frac{\partial\tilde{\phi}'^k}{\partial\lambda'_{j_1\cdots j_h}}\lambda'_{j_1\cdots j_h j}+\tilde{h}\delta^k_j=0.$ (13)

Finally, we impose the galilean relativity principle for the constitutive function $F^{j_1\cdots j_{N+1}}$, i.e., that the diagram in the next page is commutative. In other words, we must have

$$F'^{j_1\cdots j_{N+1}}\Bigg(\underbrace{\sum_{n=h}^{N}\binom{n}{h}\lambda_{i_1\cdots i_h i_{h+1}\cdots i_n}v^{i_{h+1}}\cdots v^{i_n}}_{\lambda'_{i_1\cdots i_h}}\Bigg)=$$

$$\sum_{h=0}^{N+1}\binom{N+1}{h}(-1)^{N+1-h}F^{(j_1\cdots j_h}(\lambda_A)v^{j_{h+1}}\cdots v^{j_{N+1})}$$

(Note that $F'^{j_1\cdots j_{N+1}}$ can be obtained from eq. (1) with $-v^i$ instead of v^i and $N+1$ instead of n). This relation becomes an identity if calculated in $v^j=0$, so that it is equivalent to its derivative with respect to v^j, that is

$$\sum_{h=0}^{N-1}\frac{\partial F'^{j_1\cdots j_{N+1}}}{\partial\lambda'_{i_1\cdots i_h}}\sum_{n=h+1}^{N}\binom{n}{h}(n-h)\lambda_{i_1\cdots i_h j i_{h+2}\cdots i_n}v^{i_{h+2}}\cdots v^{i_n}=$$

$$\sum_{h=0}^{N}\binom{N+1}{h}(N+1-h)(-1)^{N+1-h}F^{(j_1\cdots j_h}v^{j_{h+1}}\cdots v^{j_N}\delta^{j_{N+1})j}\quad.$$

In other words, $F'^{j_1\cdots j_{N+1}}$ must satisfy the condition

$$\sum_{h=0}^{N-1}(h+1)\frac{\partial F'^{j_1\cdots j_{N+1}}}{\partial\lambda'_{i_1\cdots i_h}}\lambda'_{i_1\cdots i_h j}+(N+1)F'^{(j_1\cdots j_N}\delta^{j_{N+1})j}=0\,. \tag{14}$$

$$\begin{tikzcd}
\boxed{\lambda_A} \arrow[rr] \arrow[dd, "F^{j_1\cdots j_{N+1}}"'] & & \boxed{\lambda'_B(\lambda_A)} \arrow[d, "F'^{j_1\cdots j_{N+1}}"] \\
 & & \boxed{F'^{j_1\cdots j_{N+1}}[\lambda'_B(\lambda_A)]} \arrow[d, equal] \\
\boxed{F^{j_1\cdots j_{N+1}}(\lambda_A)} \arrow[rr] & & \boxed{\sum_{h=0}^{N+1}\binom{N+1}{h}(-1)^{N+1-h}F^{(j_1\cdots j_h}(\lambda_A)v^{j_{h+1}}\cdots v^{j_{N+1})}}
\end{tikzcd}$$

Equations (11), (13) and (14) exhaust the conditions of the galilean relativity principle. On the other hand, one could say that with the independent variables $\lambda_{i_1\cdots i_n}$ and the dependent variables $F^{i_1\cdots i_n}$, also these last ones must satisfy the galilean relativity principle. This is true, but we can easily see that this is a consequence of the other conditions. In fact, from eq. (4) we have $dh=\lambda_{i_1\cdots i_n}dF^{i_1\cdots i_n}$; from this and from eq. $(7)_1$ we have $d\tilde{h}=F^{i_1\cdots i_n}d\lambda_{i_1\cdots i_n}$, i.e.,

$$F^{i_1\cdots i_n}=\frac{\partial\tilde{h}}{\partial\lambda_{i_1\cdots i_n}}\,. \tag{15}$$

After that, we can consider eq. (10), i.e.,

$$\tilde{h}(\lambda_{i_1\cdots i_k})=\tilde{h}\left(\sum_{n=h}^{N}\binom{n}{h}\lambda_{j_1\cdots j_h j_{h+1}\cdots j_n}v^{j_{h+1}}\cdots v^{j_n}\right).$$

By taking the derivative of this relation with respect to $\lambda_{i_1\cdots i_k}$ and taking into account that $\lambda'_{j_1\cdots j_h}$ depends on it only for $h\le k$, we obtain

$$F^{i_1\cdots i_k}=\frac{\partial\tilde{h}}{\partial\lambda_{i_1\cdots i_k}}=\sum_{h=0}^{k}\frac{\partial\tilde{h}}{\partial\lambda'_{j_1\cdots j_h}}\binom{k}{h}\delta^{(i_1}_{j_1}\cdots\delta^{i_h}_{j_h}v^{i_{h+1}}\cdots v^{i_k)}=$$

$$=\sum_{h=0}^{k}\binom{k}{h}F'^{(i_1\cdots i_h}v^{i_{h+1}}\cdots v^{i_k)}\,,$$

that is the relations (1); then the galilean relativity principle on $F^{i_1\cdots i_n}$ is satisfied as a consequence of the other conditions.

We note that eqs.(11), (13) are those listed in paper [1].
We stress that, until now, we have not imposed the entropy principle! If we impose it, we also have $F^{i_1\cdots i_n k}=\dfrac{\partial\tilde{\phi}^k}{\partial\lambda_{i_1\cdots i_n}}$.

$$\tag{16}$$

Then also eq. (14) is a consequence of the other equations. In fact, eq. (13) can be rewritten as $\displaystyle\sum_{h=0}^{N-1}(h+1)\frac{\partial\tilde{\phi}'^{j_{N+1}}}{\partial\lambda'_{i_1\cdots i_h}}\lambda'_{i_1\cdots i_h j}+\tilde{h}\delta^{j_{N+1}}_j=0$,

whose derivative with respect to $\lambda'_{j_1\cdots j_N}$ is

$$\sum_{h=0}^{N-1}(h+1)\frac{\partial F'^{j_1\cdots j_{N+1}}}{\partial\lambda'_{i_1\cdots i_h}}\lambda'_{i_1\cdots i_h j}+N\frac{\partial\tilde{\phi}'^{j_{N+1}}}{\partial\lambda'_{(j_1\cdots j_{N-1}}}\delta^{j_N)}_j+F'^{j_1\cdots j_N}\delta^{j_{N+1}}_j=0\,,$$

where eqs. (15) and (16) have been used. The result is exactly eq. (14). Moreover, we see that eq. (13), by using eqs. (16) and (15), can be rewritten

as $$\sum_{h=0}^{N-1}(h+1)\frac{\partial\tilde{h}}{\partial\lambda'_{j_1\cdots j_h k}}\lambda'_{j_1\cdots j_h j}+\tilde{h}\delta^k_j=0\,. \tag{17}$$

In this way, the galilean relativity principle implies conditions (eqs. (11) and (17)) only on the function $\tilde{h}$, in the form of partial differential equations. Obviously, also the compatibility between eqs. (15) and (16) must be imposed, which we have found from the entropy principle. We conclude by noting that eqs. (10) and (12) prove the Proposition 1 of ref. [5], where they were justified in another way and also by the fact that they hold in the kinetic approach to this subject.

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