where R and S are real constants, we find what follows.

(1) For $PQ > 0$

$$\chi(\xi,\tau) = \psi_0 \sec h^2 \left\{ \sqrt{\frac{\psi_0 Q}{P}} \, (\xi - \mathrm{v}\tau) \right\}, R = \frac{v}{P}, S = \frac{1}{P}\left(\frac{v^2}{2} - PQ\psi_0\right),$$

which represent a localized pulse travelling with speed v ("bright" envelope solitons) whose width $\sqrt{P}/\psi_0\sqrt{Q}$ depends on the maximum amplitude of the wave ψ_0.

(2) For $PQ < 0$

$$\chi(\xi,\tau) = \psi_0 \tanh^2 \left\{ \sqrt{-\frac{\psi_0 Q}{P}} \, (\xi - v\tau) \right\}, R = \frac{v}{P}, S = \frac{1}{P}\left(\frac{v^2}{2} - PQ\psi_0\right),$$

which corresponds to a localized region travelling at a speed v ("dark" envelope solitons).

The sign of the product PQ also affects the linear stability analysis. If we seek a solution of the form

$$\psi(\xi,\tau) = \psi_0 \exp\left(iQ\left|\psi_0\right|^2 \tau\right) \; [1 + \varepsilon\varphi(\xi,\tau) + ...]$$

and substitute in (15), we get , negletting terms of order ε^2, the linearized equation

$$i\frac{\partial\varphi}{\partial\tau} + \frac{P}{2}\frac{\partial^2\varphi}{\partial\xi^2} + Q\left|\psi_0\right|^2 (\varphi + \varphi^*) = 0,$$

Taking the perturbation $\varphi(\xi,\tau)$ of the form

$$\varphi = c_1 \exp\left[i(K\xi - \Omega\tau)\right] + c_2 \exp\left[-i(K\xi - \Omega\tau)\right]$$

we obtain the dispersion relation

$$\Omega^2 = \frac{K^2}{4}\left(P^2K^2 - 4PQ\ \left|\psi_0\right|^2\right).$$

The wave is stable if the product PQ is negative. When PQ is positive, the condition for the instability is

$$K < K_{crit} = 2\sqrt{\frac{Q}{P}}\ \left|\psi_0\right|.$$

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THE NON-RELATIVISTIC LIMIT OF RELATIVISTIC EXTENDED THERMODYNAMICS WITH MANY MOMENTS- PART II: HOW IT INCLUDES THE MASS, MOMENTUM AND ENERGY CONSERVATION.

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In part I of this article, Borghero, Demontis and Pennisi have obtained the limits for light speed c going to infty, of the balance equations in Relativistic Extended Thermodynamics with many moments. In order to obtain independent equations, they have taken a suitable linear combination of the equations, before taking the limit. What happens with this procedure to the relativistic conservation laws of mass, momentum and energy? Obviously, they transform in their classical counterparts; but proof of this property is not easy and is treated in this part II of the article.

1. Introduction

In the paper [1], a method has been shown to obtain starting from the balance relativistic equations

$$\begin{cases} \partial_\alpha A^{\alpha\alpha_2\cdots\alpha_N} = I_N^{\alpha_2\cdots\alpha_N} \\ \\ \partial_\alpha B^{\alpha\alpha_2\cdots\alpha_M} = I_M^{\alpha_2\cdots\alpha_M} \quad \text{with } M+N \text{ odd and } M < N \end{cases} \tag{1}$$

their non-relativistic counterparts by taking the limit when $c \to \infty$, where c is the speed of light. It is interesting to observe that we have not only transformed eqs.(1) in 3-dimensional notation in order to calculate the limit for $c \to \infty$; because, if we proceed in this way, the equations arising from eq.$(1)_2$ would be a subset of those coming from eq.$(1)_1$. In other words, subtracting from eq.$(1)_2$ some of eq.$(1)_1$ we find infinitesimals of higher order with respect to $\frac{1}{c}$. Then, we have to make a suitable linear combination with constant coefficients, but depending on c, such that its limit is finite and gives independent equations. There are various linear combinations that satisfy our requirements, but their limits are the same. For example, the linear combination that we have chosen in [1] (in the case $N = 3$, $M = 2$) is not the same combination used by Dreyer and Weiss in [2] (exposed also in [3]), although its limit as $c \to \infty$ is the same. We have chosen a different linear combination in order to avoid cumbersome calculations and their difficult justifications arising from the fact that our considerations are valid $\forall N$ and $\forall M$. The problem arises in verifying as the relativistic conservation laws of mass, momentum and energy are transformed in their classical counterparts, through our linear combination and its limit for $c \to \infty$. These results have been obtained here. In order to briefly describe the result, we consider the non-relativistic counterparts of eqs.(1) obtained by the method mentioned above, i.e.,

$$\begin{cases} \partial_t F^{i_1 \cdots i_s} + \partial_k F^{i_1 \cdots i_s k} = P^{i_1 \cdots i_s} & \text{for } 0 \le s \le N-1 \\ \partial_t F^{i_1 \cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} + & \\ \qquad + \partial_k F^{i_1 \cdots i_r k e_1 e_1 \cdots} = Q^{i_1 \cdots i_r} & \text{for } 0 \le r \le M-1 \end{cases} \tag{2}$$

If we start considering only eqs. $(1)_1$ with N even, we obtain only the equation $(2)_1$ with $\lim_{c \to \infty} P = 0$ (mass conservation) and $\lim_{c \to \infty} P^{i_1} = 0$ (momentum conservation), but losing energy conservation.
Instead, if we consider also eq. $(1)_2$, obviously for M odd, we prove, in this paper, that P^{ii} is infinitesimal, obtaining in this way energy conservation. Similarly, if we consider only eq. $(1)_1$ with N even, we obtain only eq. $(2)_1$ with $\lim_{c \to \infty} P = 0$ (mass conservation), but losing momentum and energy conservation. The presence of eq. $(1)_2$ with M odd affects also the productions in eq. $(1)_1$: we will see that, always as a consequence of eq. $(1)_2$, also P^{i_1} and P^{ii} are infinitesimal and by this fact we obtain the momentum and energy conservation. Thus, in a relativistic approach, eqs. $(1)_1$ and $(1)_2$ cannot be neglected.

2. The case with N odd and M even

Obviously, in this case is included the 14-moments one. The maximal trace of eq. $(1)_1$ gives the mass conservation law; let us express it in terms of the

tensor $p^{i_1 \cdots i_s}$, by using also the notation of paper [1]:

$$\begin{aligned} 0 &= I_N^{\alpha_2 \ldots \alpha_N} g_{\alpha_2\alpha_3} \cdots g_{\alpha_{N-1}\alpha_N} = \\ &= I_N^{\alpha_2 \ldots \alpha_N} (h_{\alpha_2\alpha_3} - t_{\alpha_2} t_{\alpha_3}) \ldots (h_{\alpha_{N-1}\alpha_N} - t_{\alpha_{N-1}} t_{\alpha_N}) = \\ &= \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h I_N^{\alpha_2 \ldots \alpha_N} t_{\alpha_2} t_{\alpha_3} \ldots t_{\alpha_{2h}} t_{\alpha_{2h+1}} h_{\alpha_{2h+2}\alpha_{2h+3}} \ldots h_{\alpha_{N-1}\alpha_N} = \\ &= \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h I_N^{0 \ldots 0 e_1 e_1 \ldots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} = \\ & \sum_{h=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{h} (-1)^h m_0^{N+2} c^{2h-1} P^{e_1 e_1 \ldots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} \end{aligned} \tag{3}$$

which can be multiplied by c^{2-N} and gives

$$P = \sum_{h=0}^{\frac{N-3}{2}} \binom{\frac{N-1}{2}}{h} (-1)^{\frac{h+N+1}{2}} c^{2h-N+1} P^{e_1 e_1 \ldots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} \tag{4}$$

whose non-relativistic limit is $$\lim_{c \to \infty} P = 0\,, \tag{5}$$

which is the mass conservation law for system (2). Similarly, the maximal trace of eq. $(1)_2$ gives momentum and energy conservation in the relativistic context. It reads: $0 = I_M^{\alpha_2 \ldots \alpha_M} g_{\alpha_3\alpha_4} \cdots g_{\alpha_{M-1}\alpha_M}$ which, with calculations similar to the ones above , becomes

$$0 = \sum_{h=0}^{\frac{M-2}{2}} \binom{\frac{M-2}{2}}{h} (-1)^h I_M^{\alpha_2 \overbrace{0 \ldots 0}^{2h} e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}}$$

from which, for $\alpha_2 = 0$ and $\alpha_2 = i_1$ respectively, we obtain

$$\begin{cases} P_M = -\sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} \\ P_M^{i_1} = -\sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{i_1 e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} \end{cases} . \tag{6}$$

Let us now consider the expression of $Q^{i_1 \ldots i_r}$ in [1], with $r = 2$, and let us compute its trace, thus obtaining:

$$(-2c^2)^{-\frac{N+M-3}{2}} Q^{ee} = \frac{1}{b_2} \sum_{q=0}^{\frac{M-4}{2}} b_{q_2}(-2c^2)^{-q} P_M^{e_1 e_1 \ldots e_q e_q e_{q+1} e_{q+1}} +$$

$$+ \frac{1}{b_2} \sum_{p=0}^{\frac{N-3}{2}} a_{p_2}(-2c^2)^{-p} P^{e_1 e_1 \ldots e_p e_p e_{p+1} e_{p+1}} \tag{7}$$

whose non-relativistic limit is $0 = (b_2)^{-1}(b_{02} \overline{P}_M^{\,e_1 e_1} + a_{02} \overline{P}^{\,e_1 e_1})$, where an overlined term denotes its non-relativistic limit. By using the property $a_{02} = -b_{02}$ (see ref. [1]), we obtain

$$\overline{P}_M^{\,e_1 e_1} = \overline{P}^{\,e_1 e_1} \quad . \tag{8}$$

Note that, in the case $M = 2$, there isn't the term on the left hand side of eq. (8), so that this eq. is $\overline{P}^{\,e_1 e_1} = 0$. In other words, we have energy conservation for eq. $(2)_1$. Let us also consider the expression of $Q^{i_1 \ldots i_r}$ in [1], with $r = 0$; by writing explicitly the terms with $q = 0$, $p = 0$ and using the expressions $(6)_1$ and (4) of P_M and P we obtain:

$$\frac{1}{c^{N+M-3}} Q = -\frac{1}{b_0}\, b_{00}(-2)^{\frac{N+M-1}{2}} \sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+4-M} \cdot$$

$$\cdot P_M^{e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}} + \frac{1}{b_0} \sum_{q=1}^{\frac{M-2}{2}} b_{q0}(-2)^{\frac{N+M-1-2q}{2}} c^{2-2q} P_M^{e_1 e_1 \ldots e_q e_q} +$$

$$\frac{1}{b_0} \sum_{p=1}^{\frac{N-1}{2}} a_{p0}(-2)^{\frac{N+M-1-2p}{2}} c^{2-2p} P^{e_1 e_1 \ldots e_p e_p} + \frac{1}{b_0} a_{00}(-2)^{\frac{N+M-1}{2}} \cdot$$

$$\cdot \sum_{h=0}^{\frac{N-3}{2}} \binom{\frac{N-1}{2}}{h} (-1)^{h+\frac{N+1}{2}} c^{2h-N+3} P^{e_1 e_1 \ldots e_{\frac{N-1-2h}{2}} e_{\frac{N-1-2h}{2}}} \tag{9}$$

whose non-relativistic limit is

$$0 = -\frac{M-2}{2} \frac{1}{b_0} b_{00}(-2)^{\frac{N+M-1}{2}} (-1)^{M-3} \overline{P}_M^{\,e_1 e_1} + \frac{1}{b_0} b_{10}(-2)^{\frac{N+M-3}{2}} \overline{P}_M^{\,e_1 e_1}$$

$$+ \frac{1}{b_0} a_{10}(-2)^{\frac{N+M-3}{2}} \overline{P}^{\,e_1 e_1} + \frac{1}{b_0} a_{00}(-2)^{\frac{N+M-1}{2}} \quad \frac{N-1}{2}(-1)^{N-1} \overline{P}^{\,e_1 e_1} ,$$

which, by using eq. (8), becomes

$$0 = \left[b_{00}(M-2)(-1)^{M-3} + b_{10} + a_{10} + a_{00}(N-1)(-1)^N\right] \overline{P}_M^{\,e_1 e_1} \text{ that is}$$

$$0 = \left[(M-2)(-1)^{M-3} + b_{10} - (N-M) - b_{10} - (N-1)(-1)^N\right] \overline{P}^{\,e_1 e_1} =$$

$$= \overline{P}^{\,e_1 e_1} .$$

In this way we have obtained energy conservation for the system (2). It remains to prove momentum conservation. To this end, let us consider the expression of $Q^{i_1 \ldots i_r}$ in [1] with $r = 1$; by writing explicitly the term with $q = 0$ and using the expression $(6)_2$ of $P_M^{i_1}$, we obtain

$$Q^{i_1}(-2c)^{\frac{3-N-M}{2}} = \frac{1}{b_1} \sum_{q=1}^{\frac{M-2}{2}} b_{q1}(-2c^2)^{-q} P_M^{i_1 e_1 e_1 \ldots e_q e_q}$$

$$+ \frac{1}{b_1} \sum_{p=0}^{\frac{N-3}{2}} a_{p1}(-2c^2)^{-p} P^{i_1 e_1 e_1 \ldots e_p e_p} +$$

$$-\frac{1}{b_1} b_{01} \sum_{h=0}^{\frac{M-4}{2}} \binom{\frac{M-2}{2}}{h} (-1)^{h+\frac{M-2}{2}} c^{2h+2-M} P_M^{i_1 e_1 e_1 \ldots e_{\frac{M-2-2h}{2}} e_{\frac{M-2-2h}{2}}}$$

whose non-relativistic limit is $0 = \frac{1}{b_1} a_{01} \overline{P}^{\,i_1}$; but $a_{01} = -b_{01} = -1$, so that it remains $\overline{P}^{\,i_1} = 0$, i.e. momentum conservation for the system (2).

3. The case with N even and M odd

Eqs. (4) and (6) still hold, but after exchanging M and N, P and P_M, $P_M^{i_1}$ and P^{i_1}, i.e.,

$$P_M = \sum_{h=0}^{\frac{M-3}{2}} \binom{\frac{M-1}{2}}{h} (-1)^{h+\frac{M+1}{2}} c^{2h-M+1} P_M^{e_1 e_1 \ldots e_{\frac{M-1-2h}{2}} e_{\frac{M-1-2h}{2}}}$$

$$P = -\sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h+2-N} P^{e_1 e_1 \ldots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}}$$

$$P^{i_1} = -\sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h+2-N} P^{i_1 e_1 e_1 \ldots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}} \tag{10}$$

The non-relativistic limit of $(10)_{2,3}$ can be quickly computed and equals $\overline{P} = 0$, $\overline{P}^{i_1} = 0$, i.e., we have mass and momentum conservation for the system (2). It remains to prove energy conservation. Now the passages after eqs. (6) and until eq. (8), of the previous section, can be adapted also to the present case (there is only to substitute the upper values of q and p with $\frac{M-3}{2}$ and $\frac{N-4}{2}$ respectively), so that eq. (8) still holds in the present case. Now the expression of $Q^{i_1 \cdots i_r}$ in [1] with r=0, by exploiting the terms with q=0, p=0 and using eqs. $(10)_{1,2}$, gives

$$Q(-2c^2)^{\frac{-N-M+3}{2}} = \frac{1}{b_0} \sum_{q=1}^{\frac{M-1}{2}} b_{q0}(-2c^2)^{-q+1} P_M^{e_1 e_1 \dots e_q e_q} +$$

$$\frac{1}{b_0} \sum_{p=1}^{\frac{N-2}{2}} a_{p0}(-2c^2)^{-p+1} P^{e_1 e_1 \dots e_p e_p} -$$

$$2\frac{1}{b_0} b_{00} \sum_{h=0}^{\frac{M-3}{2}} \binom{\frac{M-1}{2}}{h} (-1)^{h+\frac{M-1}{2}} c^{2h-M+3} P_M^{e_1 e_1 \dots e_{\frac{M-1-2h}{2}} e_{\frac{M-1-2h}{2}}} +$$

$$2\frac{1}{b_0} a_{00} \sum_{h=0}^{\frac{N-4}{2}} \binom{\frac{N-2}{2}}{h} (-1)^{h+\frac{N-2}{2}} c^{2h-N+4} P^{e_1 e_1 \dots e_{\frac{N-2-2h}{2}} e_{\frac{N-2-2h}{2}}}$$

whose non-relativistic limit is

$$0 = \frac{1}{b_0}\left[b_{10} + a_{10} - 2b_{00}\frac{M-1}{2}(-1)^{M-2} + 2a_{00}\frac{N-2}{2}(-1)^{N-3}\right]\overline{P}^{\,e_1 e_1} =$$

$$= \frac{1}{b_0}\left[b_{10} - (N-M) - b_{10} + M - 1 + N - 2\right]\overline{P}^{\,e_1 e_1} = \frac{1}{b_0}[2M-3]\overline{P}^{\,e_1 e_1}$$

from which $\overline{P}^{\,e_1 e_1} = 0$, i.e. we have energy conservation for the system (2). In this way all our aims have been accomplished.

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EXISTENCE AND ENERGY CONSERVATION FOR THE BOLTZMANN EQUATION

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The paper presents a recent result by the author concerning Maxwell molecules, without any cutoff in the collision kernel, in the one-dimensional case. Conservation of energy also holds.

1. Introduction

The well-posedness of the initial value problem for the Boltzmann equation means that there is a unique nonnegative solution preserving the energy and satisfying the entropy inequality, from a positive initial datum with finite energy and entropy. However, for general initial data, it is difficult, and until now not known, whether such a well-behaved solution can be constructed globally in time. The difficulty in doing this is obviously related to the nonlinearity of the collision operator and the apparent lack of conservation laws or a priori estimates preventing the solution from becoming singular in finite time.

The existence theorem of DiPerna and Lions[8] is rightly considered as a basic result of the mathematical theory of the Boltzmann equation. Unfortunately, it is far from providing a complete theory, since there is no proof of uniqueness; in addition, there is no proof that energy is conserved and conservation of momentum can be proved only globally and not locally.

It seems rather clear that in order to achieve some progress in the study of the initial value problem for the nonlinear Boltzmann equation and prove that the typical solutions have more properties that those proved in the theorem by DiPerna and Lions, more *a priori* estimates are needed. We