for all $t \geq 0$. It is not difficult to see that there exists a range of values for the parameter α where (2.18) is satisfied by virtue of the structure of (2.7).

We can now summarise our findings in the following theorem, formulated in terms of the original state variable u.

Theorem 2.1. *Let $\alpha > 2/7$ and let the initial data satisfy the conditions*

$$\int_\Omega (u(x,y,0)-1)^2 \, dx \, dy \leq J^*,$$

where J^* is the smallest positive root of $f(J) = 0$ where $f(J)$ is as in (2.7). Assume further that the initial data satisfy the conditions (2.14) and (2.16) and that α is in the range so that (2.18) is also satisfied. Then the solution $u(x,y,t)$ of (1.1) satisfies $u(x,y,t) \geq 0$ for all $t \geq 0$ and all $(x,y) \in \Omega$, and $\|u(x,y,t)-1\|_\infty \to 0$ as $t \to +\infty$.

References

1. Protter, M.H. & Weinberger, H.F. 1984 *Maximum principles in differential equations*. Springer.
2. Peletier, L.A. & Troy, W.C., *Spatial Patterns*, Birkhauser, 2001.
3. B. Straughan, *The Energy Method, Stability and Nonlinear Convection*, second edition, Springer-Verlag, New York, 2003.
4. J.N. Flavin and S. Rionero, *Qualitative Estimates for Partial Differential Equations*, (CRC Press, Boca Raton, 1995).
5. C.R. Doering and J.D. Gibbon, *Applied Analysis of the Navier-Stokes Equations*, Cambridge University Press, (Cambridge, 1995).
6. Ilyin, A.A. 2005 Lieb-Thirring inequalities and their applications to attractors of the Navier-Stokes equations. *Mat. Sbornik*, **196**:1, 33-66 (2005); English transl. in *Sb. Math.*, **196**:1 (2005).
7. Bartuccelli, M.V., Doering, C.R., Gibbon, J.D. & Malham, S.J.A. 1993 Length Scales in Solutions of the Navier-Stokes Equations. *Nonlinearity* **6**, 549-568.
8. Bartuccelli, M.V.; Gourley, S.A.; Ilyin, A.A. Positivity and the attractor dimension in a fourth-order reaction-diffusion equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 458 (2002), no. 2022, 1431–1446.
9. Ilyin, A.A. 1998 Best Constants in Multiplicative Inequalities for Sup-Norms. *J. Lond. Math. Soc.* **58**, 84-96.

THE NON-RELATIVISTIC LIMIT OF RELATIVISTIC EXTENDED THERMODYNAMICS WITH MANY MOMENTS- PART I: THE BALANCE EQUATIONS

F. BORGHERO, F. DEMONTIS AND S. PENNISI

Dipartimento di Matematica ed Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy; e-mail: borghero@unica.it;fdemontis@unica.it;spennisi@unica.it

The non-relativistic limit of Relativistic Extended Thermodynamics with 14 moments can be found in a paper by Dreyer and Weiss, which has been widely appreciated. In particular it suggest a particular structure for the classical counterpart of the theory, in particular that developed by Kremer, instead of the previous one with 13 moments. The same thing needs to be obtained with an arbitrary but fixed number of moments, and this is the object of the present paper. Also our results predict a particular structure for the classical counterpart with many moments, and it is not the simpler one. Moreover, from the passages here involved, it is evident that the deviation from the dominant parts of the equations is of the second order with respect to $1/c$, with c the speed of light. This is interesting also for future applications; it suffices now to remember that the Maxwell equations are linear with respect to $1/c$.

1. Introduction

The above mentioned paper by Dreyer and Weiss can be found in [1] and [2]. Here we extend their methods for the case with many moments following the macroscopic approach. In this way we obtain very interesting results. For example, we see that the relativistic theory seems to suggest a particular structure for the balance equations (see eqs.(2) below) in non-relativistic extended thermodynamics. It is noteworthy that in this structure the independent variables are moments of increasing orders; the highest of these is even, as in the kinetic approach! To be more precise we consider the following balance equations for the relativistic approach

$$\partial_\alpha A^{\alpha\alpha_2\cdots\alpha_N} = I_N^{\alpha_2\cdots\alpha_N} \quad , \quad \partial_\alpha B^{\alpha\alpha_2\cdots\alpha_M} = I_M^{\alpha_2\cdots\alpha_M} \tag{1}$$

where M and N are assigned numbers. It is not restrictive to suppose that $M < N$. These equations contain also those with lower order of moments because of the trace conditions $A^{\alpha\alpha_2\cdots\alpha_{N-1}}{}_{\alpha_{N-1}} = -m_0c^2 A^{\alpha\alpha_2\cdots\alpha_{N-2}}$ and

similarly for $B^{\cdots}$, where m_0 is the rest mass and c the light speed. Consequently, in order to obtain independent equations, one among M and N has to be even and the other odd, i.e., $M+N$ is odd.
Another consequence of the trace conditions is that the maximal traces of eqs.(1) are the conservation laws of mass, momentum and energy. We will see that the non-relativistic limit of eqs.(1) has the form

$$\begin{cases} \partial_t F^{i_1\cdots i_s} + \partial_k F^{i_1\cdots i_s k} = P^{i_1\cdots i_s} \\ \partial_t F^{i_1\cdots i_r e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} + \partial_k F^{i_1\cdots i_r k e_1 e_1 \cdots} = Q^{i_1\cdots i_r} \end{cases} \tag{2}$$

for $0 \leq s \leq N-1$, $0 \leq r \leq M-1$. All of the properties above described follow easily . In particular, when $M=2$, $N=3$, eqs.(1) are the pertinent equations of the 14-moments theory of relativistic extended thermodynamics [3] and eqs. (2) are the corresponding equations for the non-relativistic approach [4]. We note that the highest order of moments, among the independent variables, is $M+N-1$, which is always even; this confirms the same property obtained by the kinetic approach in order to have integrability, i.e., that the integrals involved must be convergent.

2. Suggestion from kinetic theory.

Because the form of equations (1) is suggested from the kinetic theory of gases, it is not restrictive to deduce from this theory the orders of greatness of the moments and productions with respect to c. Meanwhile, we obtain this information also for the entropy and entropy-flux tensor h^α. In particular, we have

$$\begin{aligned} A^{\alpha_1\ldots\alpha_N} &= \int \widetilde{f}(x^\mu, p^0, p^i) p^{\alpha_1}\cdots p^{\alpha_N} \tfrac{dp^1 dp^2 dp^3}{p^0} \\ h^\alpha &= \int G[\widetilde{f}(x^\mu, p^0, p^i)] p^\alpha \tfrac{dp^1 dp^2 dp^3}{p^0} \end{aligned} \tag{3}$$

where $\widetilde{f}$ is the relativistic distribution function, $p^\mu = m_0 u^\mu \equiv (m_0\gamma(u)c, m_0\gamma(u)u^i)$ is the relativistic momentum particle, $\gamma(u) = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$ is the Lorentz factor and G a suitable function of $\widetilde{f}$.
By changing the integration variables from p^i to u^i, we see that the Jacobian of the transformation is $J = \left|\frac{\partial p^i}{\partial u^j}\right| = \left| m_0\gamma(u)\delta^{ij} + m_0\frac{\gamma^3}{c^2}u^i u^j\right| = m_0^3\gamma^5$ and the above integrals (3) transform into

$$A^{\alpha_1\ldots\alpha_s\overbrace{0\ldots0}^{N-s}} = m_0^{N+2}c^{N-s-1}\widetilde{F}_N^{i_1\ldots i_s},\ h^0 = m_0^3 h,\ h^i = \frac{m_0^3}{c}\phi^i \quad \text{with}$$

$$\widetilde{F}_N^{i_1\ldots i_s} = \int \widetilde{f}\gamma^{N+4}u^{i_1}\cdots u^{i_s} d\underline{u},\ h = \int G(\widetilde{f})\gamma^5 d\underline{u},\ \phi^i = \int G(\widetilde{f})\gamma^5 u^i d\underline{u}. \tag{4}$$

Now eqs. (1) can be written as $\frac{1}{c}\partial_t A^{0\alpha_2\ldots\alpha_N} + \partial_k A^{k\alpha_2\ldots\alpha_N} = I^{\alpha_2\ldots\alpha_N}$, which can be written for $\alpha_2\ldots\alpha_N = i_1\ldots i_s 0\ldots 0$ and becomes the first of the following equations

$$\begin{cases} \partial_t \widetilde{F}_N^{i_1\cdots i_s} + \partial_k \widetilde{F}_N^{k i_1\cdots i_s} = \widetilde{P}^{i_1\cdots i_s} \text{ for } 0 \leq s \leq N-1 \\ \partial_t \widetilde{F}_M^{i_1\cdots i_r} + \partial_k \widetilde{F}_M^{k i_1\cdots i_r} = \widetilde{P}_M^{i_1\cdots i_r} \text{ for } 0 \leq r \leq M-1 \end{cases} \tag{5}$$

with
$$\widetilde{P}^{i_1\ldots i_s} = m_0^{-N-2}c^{-N+s+2}I_N^{i_1\ldots i_s 0\ldots 0},\ \widetilde{P}_M^{i_1\ldots i_r} = m_0^{-M-2}c^{-M+r+2}I_M^{i_1\ldots i_r 0\ldots 0} \tag{6}$$

and, obviously, eq.$(5)_2$ is the counterpart of eq.$(1)_2$, where $B^{\cdots\alpha_M}$ is defined in the same way as $A^{\cdots\alpha_N}$. Similarly, the entropy law $\partial_\alpha h^\alpha = \sigma$ becomes

$$\partial_t h + \partial_k \phi^k = s = m_0^{-3}c\sigma\,. \tag{7}$$

Eqs.(5) are still relativistic, although expressed in 3-dimensional form. Their limits as $c \longrightarrow \infty$ don't give independent equations, because from eq.(4) it follows that $\lim_{c\to\infty}\widetilde{F}_N^{i_1\ldots i_r} = \lim_{c\longrightarrow\infty}\widetilde{F}_M^{i_1\ldots i_r}$. In other words, $\widetilde{F}_N^{i_1\ldots i_r} - \widetilde{F}_M^{i_1\ldots i_r}$ is higher order infinitesimal with respect to c^{-1}, so that we have to find a suitable linear combination of eqs.(5) and multiply the result by an appropriate power of c, before taking the limit. This will be done in the next section.

3. A new form for the system (5).

Let us consider the numbers

$$b_{hr} = (-1)^h\binom{m}{h}\frac{(n+m-h)!}{(n+m)!}\eta\,(N-M-2n, N-M-2n+2h-2) \tag{8}$$

where $\eta(a,b)$ denotes the product of all odd numbers between a and b if $a \leq b$, while it is 1 if $a > b$; moreover $n = \left[\frac{N-1-r}{2}\right]$, $m = \left[\frac{M-1-r}{2}\right]$. Obviously, $b_{0r} = 1$. In the appendix the following will be proved
Proposition 3.1: The numbers defined by eq. (8) satisfy the equations

$$\sum_{h=0}^{m} b_{hr}c_{n+j-h} = \delta_{j,m+1}b_r, \quad \text{for } j = 1,\ldots,m+1 \text{ with} \tag{9}$$

$$c_h = \frac{1}{h!}\eta\,(N-M-2h+2, N-M)\,, \tag{10}$$

$$b_r = (-1)^m\frac{n!}{(n+m+1)!}\frac{m!}{(n+m)!}\eta\,(N-M-2n, N-M+2m)\,. \tag{11}$$

Let us also consider the numbers

$$a_{kr} = -\sum_{h=0}^{inf\{k,[\frac{M-1-r}{2}]\}} b_{hr}c_{k-h}, \text{ for } k = 0,\ldots,[\tfrac{N-1-r}{2}]\,. \tag{12}$$

After that, let us consider the following linear combination of $\widetilde{F}_M^{i_1\cdots i_r a}$ and of $\widetilde{F}_N^{i_1\cdots i_s a}$:

$$\widetilde{F}^{i_1\cdots i_r a e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \left[\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr} \widetilde{F}_M^{i_1\cdots i_r e_1 e_1 \cdots e_q e_q a}(-2c^2)^{-q} + \right.$$

$$\left. + \sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr} \widetilde{F}_N^{i_1\cdots i_r e_1 e_1 \cdots e_p e_p a}(-2c^2)^{-p}\right] \frac{1}{b_r}\left(-2c^2\right)^{\frac{M+N-1-2r}{2}} \qquad (13)$$

where the index a has to be omitted if it is zero. Note that this tensor has $N+M-1-r > N-1$ indices (if $a=0$) so that there is no possibility of confusing it with $\widetilde{F}_N^{i_1\cdots i_r}$. The corresponding linear combination of eqs.(5) gives eqs.(2)$_2$, while eq.(2)$_1$ is eq.(5)$_1$ except that now the index N has been omitted. Obviously, we also define

$$\widetilde{Q}^{i_1\cdots i_r} = \frac{1}{b_r}\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr}(-2c^2)^{\frac{N+M-1-2r-2q}{2}} \widetilde{P}_M^{i_1\cdots i_r e_1 e_1 \cdots e_q e_q} +$$

$$+ \frac{1}{b_r}\sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr}(-2c^2)^{\frac{N+M-1-2r-2p}{2}} \widetilde{P}^{i_1\cdots i_r e_1 e_1 \cdots e_p e_p} \qquad (14)$$

where the property $[\frac{N-1-r}{2}] + [\frac{M-1-r}{2}] = \frac{N+M-3-2r}{2}$ has been used (it is a consequence of the fact that $N+M$ is odd). The interesting thing, which we now prove, is that

$$\lim_{c\to\infty} \widetilde{F}^{i_1\cdots i_r a e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \int f u^{i_1}\cdots u^{i_r} u^a (u^2)^{\frac{N+M-1-2r}{2}} d\underline{u}, \qquad (15)$$

and we indicate this limit by $F^{i_1\cdots i_r a e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}}$; moreover, u^a is 1 if $a=0$, is u^k if $a=k$ and $f = \lim_{c\to\infty} m_0^3 \widetilde{f}$, as in [2] (in the sequel the factor m_0^3 does not affect the results, so we will omit it). To prove eq.(15), we see that (13), by means of (4) gives

$$\widetilde{F}^{i_1\cdots i_r a e_1 e_1 \cdots e_{\frac{N+M-1-2r}{2}} e_{\frac{N+M-1-2r}{2}}} = \frac{1}{b_r}(-2c^2)^{\frac{N+M-1-2r}{2}} \int \widetilde{f}\gamma^{N+4} u^{i_1}\cdots u^{i_r} u^a$$

$$\cdot\left[\gamma^{M-N}\sum_{q=0}^{[\frac{M-1-r}{2}]} b_{qr}(u^2)^q(-2c^2)^{-q} + \sum_{p=0}^{[\frac{N-1-r}{2}]} a_{pr}(u^2)^p(-2c^2)^{-p}\right] d\underline{u}. \qquad (16)$$

Let us denote the expression between square brackets as $[\cdots]$; by inserting the expansion of $(\gamma)^{M-N} = \left(1 - \frac{u^2}{c^2}\right)^{\frac{N-M}{2}} = \sum_{h=0}^{\infty} \frac{1}{h!} c_h (u^2)^h (-2c^2)^{-h}$

it becomes

$$[\cdots] = \left[c_0 + c_1 u^2(-2c^2)^{-1} + \ldots + c_h (u^2)^h (-2c^2)^{-h}\right]\cdot$$

$$\left[b_{0r} + b_{1r}u^2(-2c^2)^{-1} + \cdots + b_{[\frac{M-1-r}{2}],r}\left(\frac{u^2}{-2c^2}\right)^{[\frac{M-1-r}{2}]}\right] + \sum_{k=0}^{[\frac{N-1-r}{2}]} a_{kr}(u^2)^k$$

$$\cdot(-2c^2)^{-k} = \sum_{k=0}^{\infty}\sum_{h=0}^{\inf\{k,[\frac{M-1-r}{2}]\}} b_{hr}c_{k-h}\left(\frac{u^2}{-2c^2}\right)^k + \sum_{k=0}^{[\frac{N-1-r}{2}]} a_{kr}\left(\frac{u^2}{-2c^2}\right)^k.$$

Now, the tensor for $k \leq [\frac{N-1-r}{2}]$ disappears for eq.(12), while those with $[\frac{N-1-r}{2}] + 1 \leq k \leq [\frac{N-1-r}{2}] + [\frac{M-1-r}{2}] = m+n$ disappear for eqs.(10) with $j = k-n$ (note that $1 \leq j \leq m$). It remains to consider the terms with $k = m+n+1$ and those of higher order, i.e.,

$$[\cdots] = \left(-2c^2\right)^{-m-n-1}\left[\sum_{h=0}^{m} b_{hr}c_{m+n+1-h}\left(u^2\right)^{m+n+1} + o\left(\frac{1}{c^2}\right)\right].$$

Inserting this result in eq.(16), using eq.(10) with $j = m+1$, and taking the limit as $c \longrightarrow \infty$, we obtain eq.(15). This completes the proof.

4. Appendix

In order to prove the properties (10) and (11), we first state the following

Lemma 4.1. *For every $k \in [0, m-1]$ we have $\sum_{h=0}^{m}(-1)^h\binom{m}{h}h^k = 0$.*

Proof. We proceed with the iteration method with respect to k. The property is true when $k=0$ because, in this case, the first member corresponds to $(-1+1)^m = 0$. If we assume that it is true up to a fixed integer $\overline{k} < m-2$, we have

$$\sum_{h=0}^{m}(-1)^h\binom{m}{h}h^{\overline{k}+1} = \sum_{h=1}^{m}(-1)^h\binom{m}{h}h^{\overline{k}+1} = m\sum_{h=1}^{m}(-1)^h\binom{m-1}{h-1}h^{\overline{k}} =$$

$$-m\sum_{s=0}^{m-1}(-1)^s\binom{m-1}{s}h^{\overline{k}} = 0,$$

where in the third passage we have put $h = s+1$. □

Now we consider the following functions

$$f(n,m,N-M,j) = \sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{n!}\frac{(n+j)!}{(n+j-h)!}\cdot$$

$$\cdot\eta\left(N-M-2n, N-M-2n+2h-2\right)\cdot$$

$$\cdot\eta\left(N-M-2n-2j+2h+2, N-M-2n-2j+2m\right). \qquad (17)$$

It is easy to prove the following **Proposition 4.1**: "$f(n,m,N-M,1)=0$."
Proof: We have
$f(n,m,N-M,1) = \frac{(n+1)!}{n!}\eta(N-M-2n, N-M-2n-2j+2m)\cdot$
$\cdot\sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{(n+1-h)!}$ where the factor $\sum_{h=0}^{m}(-1)^h\binom{m}{h}\frac{(n+m-h)!}{(n+1-h)!}$
is equal to zero both for the previous lemma and because of $\frac{(n+m-h)!}{(n+1-h)!}$ is a polynomial of degree $m-1$ in the variable h.
Proposition 4.2: "$f(n,1,N-M,j) = -\delta_{j,2}(N-M+2)$ for $j=1,2$."
Proof: We obtain $f(n,1,N-M,j) = (n+1)(N-M-2n-2j+2) - (N-M-2n)(n+j) = (N-M-2)(1-j)$, with easy calculations.
Proposition 4.3: "For every $j=1,\ldots,m,m+1$, we have
$f(n,m,N-M,j) = \delta_{j,m+1}\eta(N-M+2, N-M+2m)(-1)^m m!$."
We omit the proof for the sake of brevity.
Corollary: "$f(n,m,N-M,j) = (1-j)(2-j)\cdot\ldots\cdot(m-j)\cdot$
$\cdot\eta(N-M+2, N-M+2m)$".
Proof: We observe that first and second member in this equation are polynomial of degree m in j (for the first member it is consequence of the fact that $\frac{(n+j)!}{(n+j-h)!} = (n+j)(n+j-1)\ldots(n+j-h+1)$ and this expression has degree h in j while $\eta(N-M-2n-2j+2h+2, N-M-2n-2j+2m)$ has degree $m-h$ in j). Moreover these members give the same results in the $m+1$ different values $j=1,\ldots,m+1$.
Now we can prove the property 1: to this end it suffices to substitute eq.(8) in the left-hand side of eq.(9) and to use the definition (17); after that the identity
$\frac{\eta(N-M-2n-2j+2h-2,N-M)}{\eta(N-M-2n-2j+2h+2,N-M-2n-2j)} = \eta(N-M-2n-2j+2m+2, N-M)$
has to be used and the proposition 3 to be applied.

References

1. Dreyer W., Weiss W., "The classical limit of relativistic extended thermodynamics". Annales de l' Institut Henri Poincaré **45** (1986).
2. Müller I., Ruggeri T., Rational Extended Thermodynamics, second Edition, Springer Verlag, New York, Berlin Heidelberg (1998).
3. Liu I-Shih, Müller I., Ruggeri T., "Relativistic Thermodynamics of Gases", Annals of Physics **169** (1986).
4. Kremer G.M., "Extended Thermodynamics of ideal Gases with 14 fields", Annales de l' Institut Henri Poincaré **45** (1986).

HYDRODYNAMIC MODELS FOR A TWO-BAND NONZERO-TEMPERATURE QUANTUM FLUID

G. BORGIOLI*, G. FROSALI**, C. MANZINI**

Dip. di Elettronica e Telecomunicazioni, Dip. di Matematica Applicata***
Università di Firenze, Via S. Marta 3, I-50139 Firenze, Italy
E-mail: giovanni.borgioli@unifi.it

In this paper a hydrodynamic set of equations is derived from a Schrödinger-like model for the dynamics of electrons in a two-band semiconductor, via the Madelung ansatz. A diffusive scaling allows to attain a drift-diffusion formulation.

1. Introduction

Recent advances in semiconductor devices design have compelled the scientific community to provide theoretical models that take fully into account the quantum dynamics of carriers. For example the Resonant Interband Tunneling Diode (RITD[15]) is built on the quantum effect of tunneling of electrons between conduction and valence bands. Multiband models[11,12] derived from the Schrödinger equation are the starting point of a recent series of articles[3,4,6] that propose a two-band description in terms of Wigner functions. However, in the perspective of numerical simulations, quantum hydrodynamic models[5,7,10] are preferable, since they involve directly macroscopic quantities and they admit natural boundary conditions. The Madelung equations constitute the fluiddynamical equivalent of the Schrödinger equation and they are formally identical to the Euler equations for a perfect fluid at zero temperature, apart for the Bohm potential[9]. Analogously, two-band zero-temperature quantum fluiddynamical models[1,2] can be derived by applying the Madelung ansatz either to the two-band Schrödinger-like model introduced by Kane[11], or to the MEF (Multiband Envelope Function) model[13]; the latter one, at difference with the Kane model, seems to be reliable also in presence of heterostructures and impurities of the semiconductor material. Here, the derivation[2] is extended to the case when the electron ensemble is described by mixed states: Madelung-like equations for each band are recovered, coupled by "interband