AN EXACT MACROSCOPIC EXTENDED MODEL WITH MANY MOMENTS FOR ULTRARELATIVISTIC GASES

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Relativistic Extended Thermodynamics is a very interesting theory which is widely appreciated. Here its methods are applied to ultrarelativistic gases, and an arbitrary, but fixed, number of moments is considered. The kinetic approach has already been developed in literature; then, the macroscopic approach is here considered and the constitutive functions appearing in the balance equations are determined up to whatever order with respect to thermodynamical equilibrium. The results of the kinetic approach are a particular case of the present ones.

1. Introduction

Relativistic Extended Thermodynamics is a well established and appreciated physical theory (see refs. [1,2] regarding the first pioneering paper on this subject and an exhaustive description of the results which has been subsequently found). More recent results regarding the kinetic approach are described in refs. [3-6] and many interesting properties are there obtained and exposed. The macroscopic approach has been also investigated, but the exact solution of the conditions which are present in Relativistic Extended Thermodynamics, for the case of an ultrarelativistic gas with many moments, is still lacking. This gap is here filled, and it is shown that the general solution can be obtained with little, but meaningful modifications of the kinetic approach.

The balance equations of Extended Thermodynamics for an ultrarelativistic gas with many moments are

$$\partial_\alpha A^{\alpha\beta_1\cdots\beta_n} = I^{\beta_1\cdots\beta_n} \quad \text{for } n = 0, \cdots, N. \tag{1}$$

The tensors $A^{\alpha\beta_1\cdots\beta_n}$ and $I^{\beta_1\cdots\beta_n}$ are symmetric and trace-less. In particular, A^α is also indicated with V^α and denotes the particle number-particle flux vector, while $A^{\alpha\beta_1}$ is also indicated with $T^{\alpha\beta_1}$, i.e., the stress-energy-momentum tensor.

The entropy principle for the balance equations (1), thanks to Liu' s Theorem [7], amounts to assuming the existence of the Lagrange multipliers $\Sigma_{\beta_1\cdots\beta_n}$, symmetric and trace-less, and of a 4-vectorial function h'^{α} (related to the entropy - entropy flux) such that

$$A^{\alpha\alpha_1\cdots\alpha_n} = \frac{\partial h'^{\alpha}}{\partial\Sigma_{\beta_1\cdots\beta_n}} P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}\,, \tag{2}$$

$$\sum_{n=2}^{N} \Sigma_{\beta_1\cdots\beta_n} I^{\beta_1\cdots\beta_n} \geq 0\,,$$

where the Lagrange multipliers have been taken as independent variables and $P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ is the constant tensor such that, for every tensor $B^{\beta_1\cdots\beta_n}$, the new tensor $B^{\beta_1\cdots\beta_n}P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ is symmetric, trace-less and is equal to the sum of $B^{\alpha_1\cdots\alpha_n}$ and of a linear combination of its traces through constant tensorial coefficients. Its expression can be found for example in ref. [8] and reads

$$P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} = \sum_{s=0}^{[n/2]} P^n_s g^{(\alpha_1\alpha_2}\cdots g^{\alpha_{2s-1}\alpha_{2s}} g^{\alpha_{2s+1}}_{\beta_{2s+1}}\cdots g^{\alpha_n)}_{\beta_n} g_{\beta_{2s}\beta_{2s-1}}\cdots g_{\beta_2\beta_1}\,,$$

with P^n_s defined by the following recurrence formula

$$P^n_{s+1} = -\frac{1}{4}P^n_s\frac{n-2s}{n-s}\frac{n-2s-1}{s+1} \quad , \quad P^n_0 = 1\,.$$

Now, the eq. (2) show that $A^{\alpha\beta_1\cdots\beta_n}$ are known in terms of h'^{α} and their symmetry impose conditions on the representation of h'^{α}.

There are now three ways in which to proceed:

(1) Approach at a macroscopic level: It uses the representation theorems for isotropic functions to write down the most general expressions for h'^{α}, $I^{\beta_1\cdots\beta_n}$; after that, the symmetry conditions and the zero trace conditions for $(2)_1$ impose restrictions on h'^{α}, while $(2)_2$ does the same for $I^{\beta_1\cdots\beta_n}$. This approach is here used in an expansion around thermodynamical equilibrium and up to whatever order.

(2) Approach at a kinetic level: it proposes h'^{α}, except for an arbitrary single-variable function $f(x)$, as

$$h'^{\alpha} = \int F(\Sigma_0 + \Sigma_{\beta_1}p^{\beta_1} + \cdots + \Sigma_{\beta_1\cdots\beta_N}p^{\beta_1}\cdots p^{\beta_N})p^{\alpha}dP\,, \tag{3}$$

where p^α is the 4-momentum of the particle so that we have $p^\alpha p_\alpha = 0$ and $dP = \sqrt{-g}\frac{dp^1 dp^2 dp^3}{p^0}$ is the invariant element of momentum space.
It is easy to see that the expression (3) satisfies the symmetry and the trace conditions for $(2)_1$.
This expression has been obtained by Boillat and Ruggeri [5], who have studied also interesting mathematical properties of the subsequent model.

(3) A generalized kinetic approach : It is here proposed starting from an arbitrary function f of $N+1$ variables $f(X_0, X_1, \cdots, X_N)$ and defining

$$h'^\alpha = \int F(\Sigma, \Sigma_{\beta_1} p^{\beta_1}, \cdots, \Sigma_{\beta_1 \cdots \beta_N} p^{\beta_1} \cdots p^{\beta_N}) p^\alpha dP. \quad (4)$$

It is easy to see that also this expression satisfies the symmetry and the trace conditions for $(2)_1$. It is also proved that the first and third approach are equivalent, so that we can refer to the third approach also as *the macroscopic approach.*

We note that, in the kinetic approach, the distribution function at equilibrium is determined in terms of $F(\Sigma_0 + \Sigma_{\beta_1} p^{\beta_1})$ so that the distribution function outside equilibrium can be simply obtained by substituting $\Sigma_0 + \Sigma_{\beta_1} p^{\beta_1} + \cdots + \Sigma_{\beta_1 \cdots \beta_N} p^{\beta_1} \cdots p^{\beta_N}$ to $\Sigma_0 + \Sigma_{\beta_1} p^{\beta_1}$. Instead of this, the macroscopic approach (3) has $F(\Sigma_0, \Sigma_{\beta_1} p^{\beta_1}, 0, \cdots, 0)$ for the determination of the equilibrium distribution function and its knowledge doesn't determine F outside equilibrium. This is the reason of the one parameter family of single variable functions, arising from integration, which appear in the macroscopic approach. We remind also that the counterpart of this model with only 14 moments has already been treated in [9].
A similar result for the non ultra-relativistic case would be desirable, but it hasn't until been proved.
In the next section the generalized kinetic approach will be exploited ; the first approach will be considered in section 3 and proved that it is equivalent to the third approach.

2. The generalized kinetic approach

A particular general exact solution can be found as follows:
A particular case of eq. (4) is that with F an homogeneous function of

degree h_i in the variable X_i for $i = 2, \cdots, N$, i.e.,

$$F = F_{h_2,\cdots,h_N}(X_0, X_1)\,(X_2)^{h_2}\cdots(X_N)^{h_N} \;,$$

in which case eq. (4) becomes

$$h'^{\alpha} = \int F_{h_2,\cdots,h_N}(\Sigma, \Sigma_{\beta_1}p^{\beta_1})\,(\Sigma_{\gamma_1\gamma_2}p^{\gamma_1}p^{\gamma_2})^{h_2}\cdots\left(\Sigma_{\delta_1\cdots\delta_N}p^{\delta_1}\cdots p^{\delta_N}\right)^{h_N} p^{\alpha}dP =$$

$$\int F_{h_2,\cdots,h_N}(\Sigma, \Sigma_{\beta_1}p^{\beta_1})p^{A_{21}}\cdots p^{A_{2h_2}}\cdots p^{A_{i1}}\cdots p^{A_{ih_i}}\cdots p^{A_{N1}}\cdots p^{A_{Nh_N}}p^{\alpha}dP\,\cdot$$

$$\cdot\Sigma_{A_{21}}\cdots\Sigma_{A_{2h_2}}\cdots\Sigma_{A_{i1}}\cdots\Sigma_{A_{ih_i}}\cdots\Sigma_{A_{N1}}\cdots\Sigma_{A_{Nh_N}}\;, \tag{5}$$

where A_{ir} is a symbol standing for $\beta_{1r}\cdots\beta_{ir}$ and $p^{A_{ir}}$ stands for $p^{\beta_{1r}}\cdots p^{\beta_{ir}}$.

Now, if two functions h'^{α} are such that the corresponding $A^{\alpha\alpha_1\cdots\alpha_n}$ are symmetric, also their sum satisfies the same condition, so that the solution (5) can be generalized into

$$h'^{\alpha} = \sum_{M=0}^{\infty}\;\sum_{h_2=0}^{M-h_3-\cdots-h_N}\cdots\sum_{h_i=0}^{M-h_{i+1}-\cdots-h_N}\cdots\sum_{h_{N-1}=0}^{M-h_N}\;\sum_{h_N=0}^{M} \tag{6}$$

$$B_{h_2,\cdots,h_N}^{A_{21}\cdots A_{2h_2}\cdots A_{i1}\cdots A_{ih_i}\cdots A_{N1}\cdots A_{Nh_N}\,\alpha}\,.$$

$$\cdot\Sigma_{A_{21}}\cdots\Sigma_{A_{2h_2}}\cdots\Sigma_{A_{i1}}\cdots\Sigma_{A_{ih_i}}\cdots\Sigma_{A_{N1}}\cdots\Sigma_{A_{Nh_N}}\;,$$

which is expressed as sum of homogeneous terms of degree M, with respect to equilibrium, and with

$$B_{h_2,\cdots,h_N}^{A_{21}\cdots A_{2h_2}\cdots A_{i1}\cdots A_{ih_i}\cdots A_{N1}\cdots A_{Nh_N}\,\alpha} = \tag{7}$$

$$\int F_{h_2,\cdots,h_N}(\Sigma, \Sigma_{\beta}p^{\beta})p^{A_{21}}\cdots p^{A_{2h_2}}\cdots p^{A_{i1}}\cdots p^{A_{ih_i}}\cdots p^{A_{N1}}\cdots p^{A_{Nh_N}}p^{\alpha}dP\,.$$

This last tensor can be calculated by integrating; because it is of the type

$$B^{\alpha_1\cdots\alpha_p} = \int F_{h_2,\cdots,h_N}(\Sigma, \Sigma_{\beta}p^{\beta})p^{\alpha_1}\cdots p^{\alpha_p}dP\,, \tag{8}$$

we will refer to this simplier expression; by integrating as in ref. [9], we find

$$B^{\alpha_1\alpha_2\cdots\alpha_p} = \sum_{s=0}^{[p/2]}(-1)^p\binom{p}{2s}\frac{4\pi}{2s+1}\gamma^{-(p+2)}G_{h_2,\cdots,h_N}(\Sigma)\,\cdot \tag{9}$$

$$h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2s-1}\alpha_{2s}}U^{\alpha_{2s+1}}\cdots U^{\alpha_p)}\,,$$

$$\text{where} \qquad \gamma = (-\Sigma^\mu \Sigma_\mu)^{1/2} \quad , \quad U^\alpha = (-\Sigma^\mu \Sigma_\mu)^{-1/2} \Sigma^\alpha \, , \tag{10}$$

$$h^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta \, , \, G_{h_2,\cdots,h_N}(\Sigma) = \int_0^\infty F_{h_2,\cdots,h_N}(\Sigma,\sigma)\sigma^{p+1} d\sigma \, , \, \sigma = \Sigma_\alpha p^\alpha \, .$$

We note that the function $G_{h_2,\cdots,h_N}(\Sigma)$ in eq. $(10)_4$ is arbitrary; in fact, if it is given, we can define

$$F_{h_2,\cdots,h_N}(\Sigma,\sigma) = \frac{1}{(p+1)!} G_{h_2,\cdots,h_N}(\Sigma) e^{-\sigma} \, , \tag{11}$$

which yields again eq. $(10)_4$. In the next section we will prove uniqueness of the solution so far obtained.

3. The general solution of the macroscopic approach.

Let us impose the symmetry conditions and the zero trace conditions for eq. $(2)_1$; regarding the second one of these conditions we see that

(1) when $n > 1$, $A^{\alpha\alpha_1\cdots\alpha_n} g_{\alpha_i\alpha_j} = 0$ is an identity for the presence of the tensor $P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ and similarly we find zero, if we contract the index α of $A^{\alpha\alpha_1\cdots\alpha_n}$ with another of its indices, because $A^{\alpha\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_n} g_{\alpha\alpha_i} = A^{\alpha_j\alpha_1\cdots\alpha_i\cdots\alpha\cdots\alpha_n} g_{\alpha\alpha_i} = 0$, where the symmetry of $A^{\alpha\alpha_1\cdots\alpha_n}$ has been used and α_j is any other index different from α_i.

(2) when $n = 0$, there is no trace condition for eq. (2),

(3) when $n = 1$, the trace condition is

$$\frac{\partial h'^\alpha}{\partial \Sigma_{\beta_1}} g^{\alpha\beta_1} = 0 \, . \tag{12}$$

In other words, the zero trace conditions reduce to eq. (12).

Let us now consider the polynomial expansion of h'^α in the variables Σ_{A_2}, $\cdots$, Σ_{A_N} and call $h'^\alpha_{h_2,\cdots,h_N}$ the homogeneous part of h'^α of degree h_i in Σ_{A_i} for $i = 2, \cdots N$; let us see how it is restricted by the symmetry conditions

$$\frac{\partial h'^{[\alpha}}{\partial \Sigma_{\beta_1\cdots\beta_n}} P^{\alpha_1]\cdots\alpha_n}_{\beta_1\cdots\beta_n} = 0 \, , \tag{13}$$

which arises from eq. (2). Obviously, $h'^\alpha_{h_2,\cdots,h_N}$ is restricted by (13) only when we impose this condition at the order h_i in Σ_{A_i} for $i \neq n$ and at the order h_{i-1} for $i = n$. Moreover, this restrictions don't link $h'^{\alpha}_{h_2,\cdots,h_N}$ to the other orders of h'^{α}. Similarly, the equilibrium value of h'^{α}, i.e. $h'^{\alpha}_{0,\cdots,0}$, is restricted by (13) only for $n = 1$, and doesn't link it to terms of the other orders.

Then, h'^{α} is a sum of functions of the type $h'^{\alpha}_{h_2,\cdots,h_N}$, which are restricted only by eqs.(12) and (13). Therefore, we can restrict ourselves to consider such restrictions only on $h'^{\alpha}_{h_2,\cdots,h_N}$.

Now this tensor, being a polynomial, coincides with its Taylor's expansion around the values $\Sigma_{A_2} = 0, \cdots, \Sigma_{A_N} = 0$: in this way we obtain the equation (6) with

$$B_{h_2\cdots h_N}^{A_{21}\cdots A_{2h_2}\cdots A_{i1}\cdots A_{ih_i}\cdots A_{N1}\cdots A_{Nh_N}\alpha} = \frac{1}{(h_2)!}\cdots\frac{1}{(h_N)!} \tag{14}$$

$$\frac{\partial^{h_2+\cdots+h_N} h'^{\alpha}_{h_2\cdots h_N}}{\partial\Sigma_{\beta_2 1}\cdots\partial\Sigma_{\beta_{2h_2}}\cdots\partial\Sigma_{\beta_{i1}}\cdots\partial\Sigma_{\beta_{ih_i}}\cdots\partial\Sigma_{\beta_{N1}}\cdots\partial\Sigma_{\beta_{Nh_N}}}.$$

$$P^{A_2 1}_{B_2 1}\cdots P^{A_{2h_2}}_{B_{2h_2}}\cdots P^{A_{i1}}_{B_{i1}}\cdots P^{A_{ih_i}}_{B_{ih_i}}\cdots P^{A_{N1}}_{B_{N1}}\cdots P^{A_{2Nh_N}}_{B_{Nh_N}},$$

where the tensors $P^{\cdots}_{\cdots}$ take into accounts the fact that $\Sigma_{\beta_1\cdots\beta_N}$ is traceless. Now we see that in the tensor $B_{h_2\cdots h_N}^{A_{21}\cdots A_{2h_2}\cdots A_{i1}\cdots A_{ih_i}\cdots A_{N1}\cdots A_{Nh_N}\alpha}$ defined by (14) we can exchange α with any other index, as consequence of eq.(13) or of a suitable of its derivatives. It follows that it is a symmetric tensor; in fact for any couple of indices β and γ, we can exchange α and β, then β and γ, after that γ and α, i.e.:

$B^{\cdots\beta\cdots\gamma\cdots\alpha} = B^{\cdots\alpha\cdots\gamma\cdots\beta} = B^{\cdots\alpha\cdots\beta\cdots\gamma} = B^{\cdots\gamma\cdots\beta\cdots\alpha}$.

Moreover, $B^{\cdots}_{\cdots}$ is traceless because in $B^{\cdots\beta\cdots\gamma\cdots}g_{\beta\gamma}$ we may use the symmetry to carry β and γ in indices of the same tensor $P^{\cdots}_{\cdots}$ and, after that the trace is zero. At the end we find the eq. (6) with $B_{h_2,\cdots,h_N}{}^{A_{21}\cdots A_{2h_2}\cdots A_{i1}\cdots A_{ih_i}\cdots A_{N1}\cdots A_{Nh_N}\alpha}$ symmetric and traceless; moreover, from eq.(13) with $n = 1$ and eq.(12) we have that also its derivative with respect to Σ_γ is symmetric and traceless. Vice versa, if these conditions are satisfied, then eq.(12) holds and also (13).

In order to simplify the notation and work with less indices, let us characterize a tensor $B^{\alpha_1\alpha_2\cdots\alpha_p}$ which is symmetric with its derivative with respect to Σ_γ, and with zero trace. The symmetry of $B^{\alpha_1\alpha_2\cdots\alpha_p}$ shows that it has the form

$$B^{\alpha_1\cdots\alpha_p} = \sum_{S=0}^{[p/2]}\binom{p}{2S}\frac{1}{2S+1}g_{p,2S}(\xi\,,\gamma)h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2S-1}\alpha_{2S}}U^{\alpha_{2S+1}}\cdots U^{\alpha_p)}. \tag{15}$$

In the Appendix A of ref. [10] it has been proved that the tensor $\frac{\partial B^{\alpha_1 \cdots \alpha_p}}{\partial \lambda_{\alpha_{p+1}}}$ is symmetric iff $g_{p,2S}$ satisfies the following recurrence formula

$$g_{p,2S-2} = \frac{-1}{2S+1}\left[\gamma \frac{\partial g_{p,2S}}{\partial \gamma} + (p-2S+1) g_{p,2S}\right] . \tag{16}$$

This equation determines all the functions $g_{p,2S}$ except for that with the greatest value of S, i.e., $g_{p,2[p/2]}$.
Similarly, with the same calculations of Appendix B in ref. [10], we obtain that the condition

$$B^{\alpha_1\alpha_2\alpha_3\cdots\alpha_p} g_{\alpha_1\alpha_2} = 0$$

can be expressed as

$$g_{p,2s+2} - g_{p,2s} = 0 . \tag{17}$$

This equation, together with eq. (16), reduces to the ordinary differential equation

$$\gamma \frac{\partial g_{p,2S+2}}{\partial \gamma} + (p+2) g_{p,2S+2} = 0 ,$$

which can be integrated and becomes

$$g_{p,2S+2} = (-1)^p 4\pi G_{h_2,\cdots,h_N}(\Sigma) \gamma^{-(p+2)} ,$$

with $(-1)^p 4\pi G_{h_2,\cdots,h_N}(\Sigma)$ a constant arising from integration. After that, the other relations in eqs. (16) and (17) are satisfied.
In this way the eq. (9) of the generalized kinetic approach has been obtained with $G_{h_2,\cdots,h_N}$ an arbitray function of Σ.
It remains to impose the condition

$$\frac{\partial B^{\alpha\alpha_1\cdots}_{h_2\cdots h_N}}{\partial \Sigma_\gamma} g_{\alpha\gamma} = 0 . \tag{18}$$

But, when at least one of $h_2, \cdots, h_N$ isn't zero, the tensor $B^{\alpha\alpha_1\cdots}_{h_2\cdots h_N}$ has at least another index which we have called α_1; moreover, γ and α_1 can be exchanged because we have already imposed the symmetry condition. Therefore, eq.(18) becomes $\frac{\partial B^{\alpha\gamma\cdots}_{h_2\cdots h_N}}{\partial \Sigma_{\alpha_1}} g_{\alpha\gamma} = 0$ which is an identity, because $B^{\alpha\gamma\cdots}_{h_2\cdots h_N}$ is traceless. Then, eq.(18) has to be imposed only for $h_2 = \cdots = h_N = 0$ and becomes

$$\frac{\partial h'^{\alpha}_{eq}}{\partial \Sigma_\gamma} g_{\alpha\gamma} = 0 , \tag{19}$$

where h'^{α}_{eq} denotes the value of h'^{α} at equilibrium. Now, for the representation theorems, we have

$$h'^{\alpha}_{eq} = h_0(\Sigma, \gamma)U^{\alpha}\,, \text{ so that eq. (19) becomes } \quad \gamma\frac{\partial h_0}{\partial \gamma} + 3h_0 = 0\,,$$

i.e., $h_0 = \gamma^{-3}H_0(\Sigma)$, i.e., $h'^{\alpha}_{eq.}$ has still the form (9) with $p = 1$ and $G_0(\Sigma) = \frac{-H_0}{4\pi}$. This completes the proof that the macroscopic and the generalized kinetic approach give the same result.

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