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WAVE OPERATORS FOR DEFOCUSING MATRIX ZAKHAROV-SHABAT SYSTEMS WITH POTENTIALS NONVANISHING AT INFINITY

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Abstract. In this article we prove that the wave operators describing the direct scattering of the defocusing matrix Zakharov-Shabat system with potentials having distinct nonzero values with the same modulus at $\pm\infty$ exist, are asymptotically complete, and lead to a unitary scattering operator. We also prove that the free Hamiltonian operator is absolutely continuous.

1. Introduction. In this article we develop in part the direct scattering theory for the defocusing matrix Zakharov-Shabat system

$$iJ\frac{\partial X}{\partial x}(k,x) - V(x)X(k,x) = kX(k,x), \qquad x \in \mathbb{R}, \tag{1.1}$$

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where

$$J = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & -I_n \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0 & i\boldsymbol{q}(x) \\ -i\boldsymbol{q}(x)^\dagger & 0_{n\times n} \end{pmatrix}, \tag{1.2}$$

the $1\times n$ matrix $\boldsymbol{q}(x)$ is the potential, I_p stands for the $p\times p$ identity matrix, and k is a spectral parameter. The dagger denotes the matrix conjugate transpose and the asterisk (scalar) complex conjugation. We assume that

$$\boldsymbol{q}(x) \sim \boldsymbol{q}_\pm = e^{i\theta_\pm}\boldsymbol{q}_0, \qquad x \to \pm\infty, \tag{1.3}$$

where $\boldsymbol{q}_0$ is a fixed nontrivial row vector. We also assume that

$$\boldsymbol{q}(x) - \boldsymbol{q}_\pm$$

has its entries in $L^1(\mathbb{R}^\pm)$. Putting $q_0 \stackrel{\text{def}}{=} \|\boldsymbol{q}_0\| > 0$, the Zakharov-Shabat operator $iJ(d/dx) - V$ is a selfadjoint operator on the direct sum $L^2(\mathbb{R})^{(n+1)\times 1}$ of $n+1$ copies of $L^2(\mathbb{R})$.

Equation (1.1), with $n = 1$ and nonvanishing boundary conditions, was first studied by Zakharov and Shabat [21]. Various improvements to their results soon followed [10, 11, 7, 14, 2, 4, 3]. The standard source on the $n = 1$ case is the Faddeev-Takhtajan book [6], where a Hamiltonian framework is adopted consistently. For $n \geq 2$ the $1+n$ problem with nonvanishing boundary conditions has not been studied as much as the $1+1$ problem. Here the main results were obtained by Gerdzhikov and Kulish [8] and Prinari, Ablowitz, and Biondini [16] for $n = 2$, though there still are substantial problems in identifying the domains of analyticity of Jost solutions in order to pass to a Riemann-Hilbert problem and Marchenko integral equation when solving the inverse scattering problem. For $n = 1$ such results are basically known [15, 8, 18].

The matrix Zakharov-Shabat system with vanishing boundary conditions has its continuous spectrum for $\lambda \in \mathbb{R}$. In the defocusing case the selfadjointness of the problem precludes the occurrence of other spectrum. We then end up deriving Riemann-Hilbert problems relating functions analytic in the upper half complex λ-plane to functions analytic in the lower half complex λ-plane, coupled by a (modified) scattering matrix. Under nonvanishing boundary conditions, however, the situation complicates considerably. In the $1+1$ case the continuous spectrum consists of $(-\infty, -q_0] \cup [q_0, +\infty)$, like for the Dirac equation, with uniform spectral multiplicity 2. Selfadjointness still allows the occurrence of isolated real and simple eigenvalues in the spectral gap $(-q_0, q_0)$. In the $1+n$ case (with $n \geq 2$) the continuous spectrum consists of a two-fold layer along $(-\infty, -q_0] \cup [q_0, +\infty)$ and an $(n-1)$-fold layer along the full real line.

The main topic of this article is to prove the existence and the asymptotic completeness of the Moeller wave operators defined by

$$W_\pm = \lim_{t\to\pm\infty} P_{\mathrm{ac}}(H)e^{itH}e^{-itH_0}, \qquad Z_\pm = \lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_{\mathrm{ac}}(H), \tag{1.4}$$

where $H_0 = iJ(d/dx) - V_f$ is the free Hamiltonian for some steplike potential V_f, $H = iJ(d/dx) - V$ is the (full) Hamiltonian, and $P_{\mathrm{ac}}(H)$ is the orthogonal projection of $L^2(\mathbb{R})^{(n+1)\times 1}$ onto the absolutely continuous subspace of H. Here we observe that H_0 and H are selfadjoint on $L^2(\mathbb{R})^{(n+1)\times 1}$. Then the wave operators induce a unitary equivalence between the absolutely continuous part of H and the free Hamiltonian H_0. Further, the scattering operator

$$S = Z_+W_-$$

is proven to be a unitary operator commuting with H_0.

Historically wave operators have been introduced in the time dependent scattering theory of the Schrödinger and Dirac equations [9, 17, 20, 19]. Only recently they have been applied to the matrix Zakharov-Shabat systems with L^1-potentials ([5] in the defocusing case; [13] in general), where the scattering matrix prevailing in the direct and inverse scattering theory and introduced there in an ad hoc manner was shown to be unitarily equivalent to the scattering operator from time dependent scattering theory. The unitary equivalence (in fact, the Fourier transform map) turned out to diagonalize H_0 and S simultaneously.

In this article we introduce a unitary transformation $\boldsymbol{U}$ which maps the original defocusing matrix Zakharov-Shabat system (1.1) to a new equation

$$\begin{aligned}\frac{\partial Y}{\partial x}(k,x) = &\left[\begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix} \dot{+}\, ikI_{n-1}\right] Y(k,x) \\ &+ \begin{pmatrix} 0 & \boldsymbol{Q}(x) \\ \boldsymbol{Q}(x)^\dagger & 0_{n\times n}\end{pmatrix} Y(k,x),\end{aligned} \tag{1.5}$$

where $\boldsymbol{Q}(x)$ is a row vector function whose first entry tends to $q_\pm$ as $x \to \pm\infty$ and whose other $n-1$ entries vanish as $x \to \pm\infty$. For $n = 1$ the unitary equivalence $\boldsymbol{U}$ is the identity matrix and nothing changes. For $n \geq 2$ the unitary equivalence allows one to separate the discussion of the wave operators and scattering solutions into one for the $1 + 1$ problem for a defocusing potential approaching $q_\pm \stackrel{\text{def}}{=} e^{i\theta_\pm}q_0$ as $x \to \pm\infty$ and one for an auxiliary matrix system of order $n - 1$.

Let us briefly describe the contents of the various sections. In Section 2 we introduce the conformal mapping $\lambda(k)$ of the spectral parameter and the unitary equivalence $\boldsymbol{U}$, compute the resolvent of the free Hamiltonian H_0, and use the result to diagonalize H_0. In Section 3 we prove the existence and asymptotic completeness of the wave operators (1.4) and the unitarity of the scattering operator.

In this article we denote the domain, image, and null space of a linear operator T by $\mathcal{D}(T)$, $\operatorname{Im} T$, and $\operatorname{Ker} T$, respectively.

2. Preliminaries. In this section we apply a unitary equivalence to the matrix Zakharov-Shabat system to arrive at the modified matrix Zakharov-Shabat system (1.5). We then go on to study the spectral properties of the corresponding free Hamiltonian. We start by introducing a conformal mapping of the spectral parameter.

2.1. Conformal mapping. By $\mathbb{K}$ we consider the Riemann surface consisting of two copies of the complex k-plane cut along $(-\infty,-q_0]\cup[q_0,+\infty)$, one called the physical sheet and the other the unphysical sheet, where the upper/lower edge of $[q_0,+\infty)$ of the physical sheet is glued to the lower/upper edge of $(-\infty,-q_0]$ of the unphysical sheet and the lower/upper edge of $[q_0,+\infty)$ of the physical sheet is glued to the upper/lower edge of $(-\infty,-q_0]$ of the unphysical sheet. We shall denote the physical sheet (without the branch cut limits) by $\mathbb{K}^+$ and the unphysical sheet (without the branch cut limits) by $\mathbb{K}^-$. The physical and unphysical sheets including the branch cut limits are denoted by $\overline{\mathbb{K}^+}$ and $\overline{\mathbb{K}^-}$, respectively.

For $k\in\overline{\mathbb{K}^+}$ we define the conformal map $\lambda=\lambda(k)=\sqrt{k^2-q_0^2}$ as follows [1, Sec. 2.3]:

$$\begin{cases} k-q_0=r_1e^{i\theta_1}, & 0\le\theta_1<2\pi,\\ k+q_0=r_2e^{i\theta_2}, & -\pi\le\theta_2<\pi, \end{cases}$$

where $r_1=|k-q_0|$ and $r_2=|k+q_0|$. We then define

$$\lambda(k)=\sqrt{r_1r_2}\,e^{i(\theta_1+\theta_2)/2}.$$

Thus, the argument $\theta=\frac{1}{2}(\theta_1+\theta_2)$ of $\lambda(k)$ varies continuously between 0 and π in the upper and lower complex k-planes cut along $(-\infty,-q_0]\cup[q_0,+\infty)$. We then get the upper half complex λ-plane Λ^+, where $\operatorname{Im}(\lambda\pm k)\ge 0$. On the other hand, for $k\in\overline{\mathbb{K}^-}$ we define

$$\lambda(k)=-\sqrt{r_1r_2}\,e^{i(\theta_1+\theta_2)/2}$$

and obtain the lower half complex λ-plane Λ^-, where $\operatorname{Im}(\lambda\pm k)\le 0$. Thus the so-called physical k-sheet corresponds to the upper half complex λ-plane Λ^+ and the so-called unphysical k-sheet corresponds to the lower half complex λ-plane Λ^-. These two λ half-planes are glued together by a common real λ-line, where we write $\overline{\Lambda^\pm} = \Lambda^\pm \cup \mathbb{R}$. A special role is played by $k = \pm q_0$ which correspond to $\lambda = 0$.

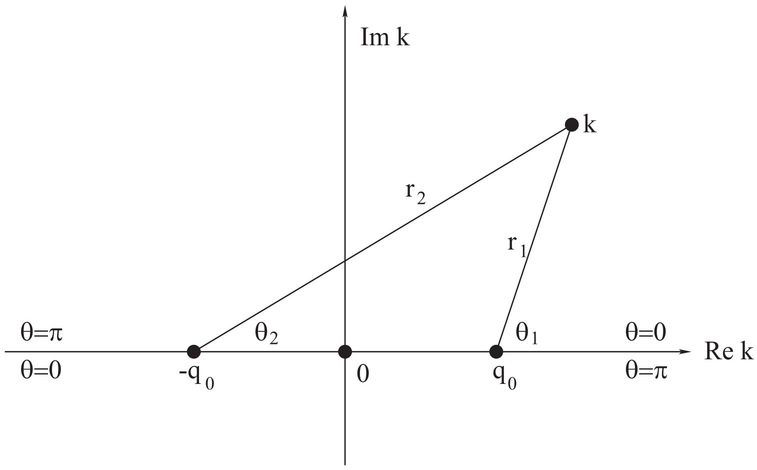

Fig. 2.1. The conformal mapping $\lambda(k)$ from the physical k-sheet cut along $(-\infty, -q_0] \cup [q_0, +\infty)$ onto the upper half complex k-plane bordered by the real k-plane. The points $k = \pm q_0$ are mapped into $\lambda = 0$

In the sequel the variable k will always be a point of the Riemann surface $\mathbb{K}$ consisting of the physical and unphysical k-sheets glued together in the appropriate way. The variable λ will always belong to the complex λ-plane Λ thought of the union of the upper and lower half complex λ-plane glued together along the real λ-line.

Because $\lambda^2 - k^2 = -q_0^2 \in \mathbb{R}$, it is clear that the conformal mapping transforms the real plus imaginary axes into the real plus imaginary axes. In particular, $k \in (-q_0, q_0) \subset \mathbb{K}^+$ is transformed into $\lambda \in (0, iq_0] \subset \Lambda^+$ and $k \in (-q_0, q_0) \subset \mathbb{K}^-$ is transformed into $\lambda \in [-iq_0, 0) \subset \Lambda^-$. Further, $k \in i\mathbb{R} \subset \mathbb{K}^+$ is transformed into $\lambda \in [iq_0, +i\infty) \subset \Lambda^+$ and $k \in i\mathbb{R} \subset \mathbb{K}^-$ is transformed into $\lambda \in (-i\infty, -q_0] \subset \Lambda^-$.

Let us now look into the transformations of the eight quadrants of the Riemann surface $\mathbb{K}$ (four in $\mathbb{K}^+$ and four in $\mathbb{K}^-$) into the four quadrants of the complex λ-plane Λ, where we number the quadrants in the usual way by the upper roman numerals I, II, III and IV. In the penultimate and ultimate columns of the table below we indicate the respective limits of $\lambda + k$ and $\lambda - k$ as $|k| \to +\infty$ (or, equivalently, as $|\lambda| \to +\infty$). Since $(\lambda + k)(\lambda - k) = -q_0^2 \neq 0$, one of the limits is infinity and the other is zero.

$\mathbb{K}^+$	I	Λ^+	I	∞	0	$\mathbb{K}^-$	I	Λ^-	III	0	∞
$\mathbb{K}^+$	II	Λ^+	II	∞	0	$\mathbb{K}^-$	II	Λ^-	IV	0	∞
$\mathbb{K}^+$	III	Λ^+	I	0	∞	$\mathbb{K}^-$	III	Λ^-	III	∞	0
$\mathbb{K}^+$	IV	Λ^+	II	0	∞	$\mathbb{K}^-$	IV	Λ^-	IV	∞	0

The signs of λ and k agree for $k \in \mathbb{K}^+$ on the upper edges and for $k \in \mathbb{K}^-$ on the lower edges of the cuts; these signs differ for $k \in \mathbb{K}^+$ on the lower edges and $k \in \mathbb{K}^-$ on the upper edges of the cuts. Moreover,

$$\lambda \pm k = \lambda \left[1 \pm \sqrt{1 + \frac{q_0^2}{\lambda^2}} \right] = \begin{cases} 2\lambda + (q_0^2/2\lambda) + O(\lambda^{-3}), & \infty \text{ in third column,} \\ \pm(q_0^2/2\lambda) + O(\lambda^{-3}), & 0 \text{ in third column,} \end{cases}$$

where the three $\pm$ vary independently but in accordance with the third columns of the above two tables.

2.2. Global unitary equivalence. Now consider, for $n \geq 2$, a fixed orthonormal set of row vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ such that $\boldsymbol{q}_0 \boldsymbol{v}_j^\dagger = 0$ $(j = 1, \ldots, n-1)$. Then

$$\boldsymbol{e}_0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0_{n\times 1} \end{pmatrix}, \ \boldsymbol{e}_q \stackrel{\text{def}}{=} \frac{1}{q_0} \begin{pmatrix} 0 \\ \boldsymbol{q}_0^\dagger \end{pmatrix}, \ \boldsymbol{e}_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \boldsymbol{v}_1^\dagger \end{pmatrix}, \ldots, \ \boldsymbol{e}_{n-1} \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \boldsymbol{v}_{n-1}^\dagger \end{pmatrix},$$

is an orthonormal basis of $\mathbb{C}^{n+1}$. Now write the matrix Zakharov-Shabat system (1.1) in the form

$$\frac{\partial X}{\partial x}(k, x) = \underbrace{\begin{pmatrix} -ik & \boldsymbol{q}_\pm \\ \boldsymbol{q}_\pm^\dagger & ikI_n \end{pmatrix}}_{=\boldsymbol{E}_\pm(k)} X(k, x) + \begin{pmatrix} 0 & \boldsymbol{q}(x) - \boldsymbol{q}_\pm \\ \boldsymbol{q}(x)^\dagger - \boldsymbol{q}_\pm^\dagger & 0_{n\times n} \end{pmatrix} X(k, x), \tag{2.1}$$

where, for $j = 1, 2, \ldots, n-1$,

$$\begin{cases} \boldsymbol{E}_\pm(k)\boldsymbol{e}_0 = -ik\boldsymbol{e}_0 + q_\pm \boldsymbol{e}_q, \\ \boldsymbol{E}_\pm(k)\boldsymbol{e}_q = q_\pm^* \boldsymbol{e}_0 + ik\boldsymbol{e}_q, \\ \boldsymbol{E}_\pm(k)\boldsymbol{e}_j = ik\boldsymbol{e}_j. \end{cases}$$

Put

$$X(k, x) = \boldsymbol{U} Y(k, x), \tag{2.2}$$

where

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{e}_0 & \boldsymbol{e}_q & \boldsymbol{e}_1 & \dots & \boldsymbol{e}_{n-1} \end{pmatrix}.$$

Then $J\boldsymbol{U} = \boldsymbol{U}J$ and

$$\boldsymbol{E}_\pm(k)\boldsymbol{U} = \boldsymbol{U} \begin{pmatrix} -ik & q_\pm & 0_{1\times(n-1)} \\ q_\pm^* & ik & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & ikI_{n-1} \end{pmatrix}.$$

As a result,

$$\begin{aligned} \frac{\partial Y}{\partial x}(k,x) &= \begin{pmatrix} -ik & q_\pm & 0_{1\times(n-1)} \\ q_\pm^* & ik & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & ikI_{n-1} \end{pmatrix} Y(k,x) \\ &\quad + \boldsymbol{U}^\dagger \begin{pmatrix} 0 & \boldsymbol{q}(x) - \boldsymbol{q}_\pm \\ \boldsymbol{q}(x)^\dagger - \boldsymbol{q}_\pm^\dagger & 0_{n\times n} \end{pmatrix} \boldsymbol{U} Y(k,x) \\ &= \left[\begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix} \dot{+}\, ikI_{n-1} \right] Y(k,x) \\ &\quad + \begin{pmatrix} 0 & \delta_{p,m}(x) & \boldsymbol{q}_v(x) \\ \delta_{p,m}(x)^* & 0 & 0_{1\times(n-1)} \\ \boldsymbol{q}_v(x)^\dagger & 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix} Y(k,x), \end{aligned} \tag{2.3}$$

where

$$\delta_p(x) = \frac{\boldsymbol{q}(x)\boldsymbol{q}_0^\dagger - e^{i\theta_+}q_0^2}{q_0} = \frac{\boldsymbol{q}(x)\boldsymbol{q}_0^\dagger}{q_0} - q_+,$$
$$\delta_m(x) = \frac{\boldsymbol{q}(x)\boldsymbol{q}_0^\dagger - e^{i\theta_-}q_0^2}{q_0} = \frac{\boldsymbol{q}(x)\boldsymbol{q}_0^\dagger}{q_0} - q_-,$$

has its entries in $L^1(\mathbb{R}^+)$ and $L^1(\mathbb{R}^-)$, respectively, and

$$\boldsymbol{q}_v(x) = \begin{pmatrix} \boldsymbol{q}(x)\boldsymbol{v}_1^\dagger & \dots & \boldsymbol{q}(x)\boldsymbol{v}_{n-1}^\dagger \end{pmatrix}$$

has its entries in $L^1(\mathbb{R})$. We have therefore converted the defocusing matrix Zakharov-Shabat system (1.1) into the equivalent $\pm$ pair of matrix Zakharov-Shabat-like systems (2.3).

In (2.3) the free Hamiltonian operators from the right and the left

$$\boldsymbol{A}_p(k) = \boldsymbol{A}_p^0(k) \dotplus ikI_{n-1} \stackrel{\text{def}}{=} \begin{pmatrix} -ik & q_+ \\ q_+^* & ik \end{pmatrix} \dotplus ikI_{n-1}, \tag{2.4a}$$

$$\boldsymbol{A}_m(k) = \boldsymbol{A}_m^0(k) \dotplus ikI_{n-1} \stackrel{\text{def}}{=} \begin{pmatrix} -ik & q_- \\ q_-^* & ik \end{pmatrix} \dotplus ikI_{n-1}, \tag{2.4b}$$

depend on the separate limits $\boldsymbol{q}_\pm$. Equation (2.3) can then be written as

$$\frac{\partial Y}{\partial x}(k,x) = \boldsymbol{A}_p(k)Y(k,x) + \underbrace{\begin{pmatrix} 0 & \boldsymbol{w}_p(x) \\ \boldsymbol{w}_p(x)^\dagger & 0_{n\times n} \end{pmatrix}}_{=\boldsymbol{V}_p(x)} Y(k,x), \tag{2.5a}$$

$$\frac{\partial Y}{\partial x}(k,x) = \boldsymbol{A}_m(k)Y(k,x) + \underbrace{\begin{pmatrix} 0 & \boldsymbol{w}_m(x) \\ \boldsymbol{w}_m(x)^\dagger & 0_{n\times n} \end{pmatrix}}_{=\boldsymbol{V}_m(x)} Y(k,x), \tag{2.5b}$$

where

$$\boldsymbol{w}_p(x) = \begin{pmatrix} \delta_p(x) & \boldsymbol{q}_v(x) \end{pmatrix}, \qquad \boldsymbol{w}_m(x) = \begin{pmatrix} \delta_m(x) & \boldsymbol{q}_v(x) \end{pmatrix}.$$

Now observe that either of $\boldsymbol{A}_p(k)$ and $\boldsymbol{A}_m(k)$ has the simple eigenvalues $\pm i\lambda(k)$ and the eigenvalue ik of multiplicity $n-1$. The groups generated by $\boldsymbol{A}_p(k)$ and $\boldsymbol{A}_m(k)$ are bounded if and only if $k > q_0$ or $k < -q_0$. In that case we have

$$e^{z\boldsymbol{A}_p(k)} = \begin{pmatrix} \cos(\lambda z) - ik\dfrac{\sin(\lambda z)}{\lambda} & q_+\dfrac{\sin(\lambda z)}{\lambda} \\ q_+^*\dfrac{\sin(\lambda z)}{\lambda} & \cos(\lambda z) + ik\dfrac{\sin(\lambda z)}{\lambda} \end{pmatrix} \dotplus e^{ikz}I_{n-1}, \tag{2.6a}$$

$$e^{z\boldsymbol{A}_m(k)} = \begin{pmatrix} \cos(\lambda z) - ik\dfrac{\sin(\lambda z)}{\lambda} & q_-\dfrac{\sin(\lambda z)}{\lambda} \\ q_-^*\dfrac{\sin(\lambda z)}{\lambda} & \cos(\lambda z) + ik\dfrac{\sin(\lambda z)}{\lambda} \end{pmatrix} \dotplus e^{ikz}I_{n-1}. \tag{2.6b}$$

Observe that (2.6) does not change if we replace λ with $-\lambda$.

For $k \notin (-\infty, -q_0] \cup [q_0, \infty)$ [if $n = 1$] or $k \notin \mathbb{R}$ [if $n \geq 2$] the matrices $\boldsymbol{A}_p(k)$ and $\boldsymbol{A}_m(k)$ do not have imaginary eigenvalues. For such k we compute the resolvents of $\boldsymbol{A}_p^0(k)$ and $\boldsymbol{A}_m^0(k)$:

$$(\zeta I_2 - \boldsymbol{A}_p^0(k))^{-1} = \frac{1}{\zeta - i\lambda}\frac{1}{2\lambda}\begin{pmatrix}\lambda - k & -iq_+\\ -iq_+^* & \lambda + k\end{pmatrix} + \frac{1}{\zeta + i\lambda}\frac{1}{2\lambda}\begin{pmatrix}\lambda + k & iq_+\\ iq_+^* & \lambda - k\end{pmatrix},$$

$$(\zeta I_2 - \boldsymbol{A}_m^0(k))^{-1} = \frac{1}{\zeta - i\lambda}\frac{1}{2\lambda}\begin{pmatrix}\lambda - k & -iq_-\\ -iq_-^* & \lambda + k\end{pmatrix} + \frac{1}{\zeta + i\lambda}\frac{1}{2\lambda}\begin{pmatrix}\lambda + k & iq_-\\ iq_-^* & \lambda - k\end{pmatrix},$$

where $\zeta \neq \pm i\lambda$. Letting $P_{p,+}^0(k)$ and $P_{p,-}^0(k)$ stand for the spectral projections of $\boldsymbol{A}_p^0(k)$ corresponding to its eigenvalues in the right and left half-planes, respectively, we get, for $k \in \mathbb{K}^+$ and hence $\operatorname{Im}\lambda > 0$,

$$P_{p,+}^0(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda + k & iq_+\\ iq_+^* & \lambda - k\end{pmatrix}, \quad P_{p,-}^0(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda - k & -iq_+\\ -iq_+^* & \lambda + k\end{pmatrix},$$

and similarly for $\boldsymbol{A}_m^0(k)$. This implies that

$$e^{z\boldsymbol{A}_p^0(k)}P_{p,\pm}^0(k) = e^{\mp i\lambda z}P_{p,\pm}^0(k), \qquad e^{z\boldsymbol{A}_m^0(k)}P_{m,\pm}^0(k) = e^{\mp i\lambda z}P_{m,\pm}^0(k).$$

Instead, if we consider $k \in \mathbb{K}^-$ and hence $\operatorname{Im}\lambda < 0$, we get

$$P_{p,+}^0(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda - k & -iq_+\\ -iq_+^* & \lambda + k\end{pmatrix}, \quad P_{p,-}^0(k) = \frac{1}{2\lambda}\begin{pmatrix}\lambda + k & iq_-\\ iq_-^* & \lambda - k\end{pmatrix},$$

which implies that

$$e^{z\boldsymbol{A}_p^0(k)}P_{p,\pm}^0(k) = e^{\pm i\lambda z}P_{p,\pm}^0(k), \qquad e^{z\boldsymbol{A}_m^0(k)}P_{m,\pm}^0(k) = e^{\pm i\lambda z}P_{m,\pm}^0(k).$$

2.3. Resolvent of the free Hamiltonian. In this subsection we compute the resolvent of the free Hamiltonian $H_0 = iJ(d/dx) - V_f$, where $\boldsymbol{q}_f(x) = \boldsymbol{q}_+$ for $x > x_0$ and $\boldsymbol{q}_f(x) = \boldsymbol{q}_-$ for $x < x_0$. The free Hamiltonians are unitarily equivalent by means of a similarity translating the point x_0. We also prove the absolute continuity of H_0.

Theorem 2.1. *Given an inhomogeneous term $F(x)$ with entries in $L^2(\mathbb{R})$ and $k \notin (-\infty, -q_0] \cup [q_0, \infty)$ for $n = 1$ and $k \notin \mathbb{R}$ for $n \geq 2$, the solution of the differential system*

(2.7a) $$\frac{\partial Y}{\partial x} = \boldsymbol{A}_p(k)Y(x) + F(x), \qquad x > x_0,$$

(2.7b) $$\frac{\partial Y}{\partial x} = \boldsymbol{A}_m(k)Y(x) + F(x), \qquad x < x_0,$$

which has its components in $L^2(\mathbb{R})$ *and is continuous in* $x = x_0$*, is given by*

$$Y(x) = \int_{-\infty}^{\infty} dy\, E(x, y; k) F(y), \tag{2.8}$$

where the Green's function $E(x, y; k)$ *is given by*

$$E(x,y;k) = \begin{cases} e^{(x-y)\boldsymbol{A}_p(k)} P_{p,-}(k) \\ +e^{(x-x_0)\boldsymbol{A}_p(k)} \boldsymbol{V}_{\#}(k)^{-1} P_{m,-}(k) e^{-(y-x_0)\boldsymbol{A}_p(k)} P_{p,+}(k), & x > y > x_0, \\ -e^{-(y-x)\boldsymbol{A}_p(k)} P_{p,+}(k) \\ +e^{(x-x_0)\boldsymbol{A}_p(k)} \boldsymbol{V}_{\#}(k)^{-1} P_{m,-}(k) e^{-(y-x_0)\boldsymbol{A}_p(k)} P_{p,+}(k), & y > x > x_0, \\ -e^{-(y-x)\boldsymbol{A}_m(k)} P_{m,+}(k) \\ -e^{-(x_0-x)\boldsymbol{A}_m(k)} \boldsymbol{V}(k)^{-1} P_{p,+}(k) e^{(x_0-y)\boldsymbol{A}_m(k)} P_{m,-}(k), & x < y < x_0, \\ e^{(x-y)\boldsymbol{A}_m(k)} P_{m,-}(k) \\ -e^{-(x_0-x)\boldsymbol{A}_m(k)} \boldsymbol{V}(k)^{-1} P_{p,+}(k) e^{(x_0-y)\boldsymbol{A}_m(k)} P_{m,-}(k), & y < x < x_0, \\ e^{(x-x_0)\boldsymbol{A}_p(k)} \boldsymbol{V}_{\#}(k)^{-1} e^{(x_0-y)\boldsymbol{A}_m(k)} P_{m,-}(k), & x > x_0 > y, \\ -e^{-(x_0-x)\boldsymbol{A}_m(k)} \boldsymbol{V}(k)^{-1} e^{-(y-x_0)\boldsymbol{A}_p(k)} P_{p,+}(k), & y > x_0 > x. \end{cases}$$

Here $\boldsymbol{V}(k)$ *and* $\boldsymbol{V}_{\#}(k)$ *are given by* (2.9) *below.*

As a result of the selfadjointness of H_0 we have

$$E(x, y; k) = E(y, x; k^*)^\dagger,$$

where $k \notin (-\infty, -q_0] \cup [q_0, \infty)$ for $n = 1$ and $k \notin \mathbb{R}$ for $n \geq 2$.

Proof. For such a solution $Y(x)$ to exist it is necessary and sufficient that there exists a vector Y_0 such that

$$Y(x) = e^{(x-x_0)\boldsymbol{A}_p(k)} P_{p,-}(k) Y_0 + \int_{x_0}^{x} dy\, e^{(x-y)\boldsymbol{A}_p(k)} P_{p,-}(k) F(y) - \int_{x}^{\infty} dy\, e^{(x-y)\boldsymbol{A}_p(k)} P_{p,+}(k) F(y)$$

for $x > x_0$, and

$$Y(x) = e^{-(x_0-x)\boldsymbol{A}_m(k)} P_{m,+}(k) Y_0 + \int_{-\infty}^{x} dy\, e^{(x-y)\boldsymbol{A}_m(k)} P_{m,-}(k) F(y) - \int_{x}^{x_0} dy\, e^{(x-y)\boldsymbol{A}_m(k)} P_{m,+}(k) F(y)$$

for $x < x_0$. The continuity requirement in $x = x_0$ implies that Y_0 satisfies

$$\begin{aligned} P_{p,-}(k)Y_0 - \int_{x_0}^{\infty} dy\, e^{-(y-x_0)\boldsymbol{A}_p(k)} P_{p,+}(k)F(y) \\ = P_{m,+}(k)Y_0 + \int_{-\infty}^{x_0} dy\, e^{(x_0-y)\boldsymbol{A}_m(k)} P_{m,-}(k)F(y). \end{aligned}$$

In other words,

$$\begin{aligned} P_{m,+}(k)Y_0 - P_{p,-}(k)Y_0 = &- \int_{x_0}^{\infty} dy\, e^{-(y-x_0)\boldsymbol{A}_p(k)} P_{p,+}(k)F(y) \\ &- \int_{-\infty}^{x_0} dy\, e^{(x_0-y)\boldsymbol{A}_m(k)} P_{m,-}(k)F(y), \end{aligned}$$

where the right-hand side is the sum of a vector in $\operatorname{Im} P_{m,+}(k)$ and a vector in $\operatorname{Im} P_{p,-}(k)$.

There is a unique way to recover Y_0 from $P_{m,+}(k)Y_0 - P_{p,-}(k)Y_0$ if and only if the 2×2 matrix composed of one column of $P^0_{m,+}(k)$ and one column of $P^0_{p,-}(k)$ is nonsingular:

$$\det \begin{pmatrix} iq_+ & -iq_- \\ \lambda - k & \lambda + k \end{pmatrix} = iq_+(\lambda + k) + iq_-(\lambda - k) \neq 0,$$

which is the case unless $k = \pm q_0$ (or $\lambda = 0$). As a result, under this condition we can compute Y_0 uniquely from $F(x)$. Put

$$\boldsymbol{V}(k) = P_{p,+}(k)P_{m,+}(k) + P_{p,-}(k)P_{m,-}(k), \tag{2.9a}$$

$$\boldsymbol{V}_{\#}(k) = P_{m,+}(k)P_{p,+}(k) + P_{m,-}(k)P_{p,-}(k). \tag{2.9b}$$

Then $\boldsymbol{V}(k)$ and $\boldsymbol{V}_{\#}(k)$ act as follows:

$$\begin{tikzcd} \operatorname{Im} P_{m,+}(k) \arrow[d, "\boldsymbol{V}(k)"'] \dotplus \operatorname{Im} P_{m,-}(k) \arrow[d, "\boldsymbol{V}(k)"'] = \mathbb{C}^{n+1} \arrow[d, "\boldsymbol{V}(k)"] \\ \operatorname{Im} P_{p,+}(k) \dotplus \operatorname{Im} P_{p,-}(k) = \mathbb{C}^{n+1} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Im} P_{p,+}(k) \arrow[d, "\boldsymbol{V}_{\#}(k)"'] \dotplus \operatorname{Im} P_{p,-}(k) \arrow[d, "\boldsymbol{V}_{\#}(k)"'] = \mathbb{C}^{n+1} \arrow[d, "\boldsymbol{V}_{\#}(k)"] \\ \operatorname{Im} P_{m,+}(k) \dotplus \operatorname{Im} P_{m,-}(k) = \mathbb{C}^{n+1} \end{tikzcd}$$

Moreover,

$$\det \boldsymbol{V}(k) = \det \boldsymbol{V}_{\#}(k) = 1 + \frac{q_0^2 - (q_+q_-^* + q_-q_+^*)}{2\lambda^2} \neq 0,$$

unless $\lambda = 0$. Thus from $P_{m,+}(k)Y_0 - P_{p,-}(k)Y_0 = Z_0$ we derive $P_{m,+}(k)Y_0 = \boldsymbol{V}(k)^{-1}P_{p,+}(k)Z_0$ and $P_{p,-}(k)Y_0 = -\boldsymbol{V}_\#(k)^{-1}P_{m,-}(k)Z_0$.

Equations (2.7) and (2.8) now easily lead to the expressions for the above Green's function $E(x,y;k)$. $\square$

The Green's function $E(x,y;k)$ and the matrices $\boldsymbol{V}(k)$ and $\boldsymbol{V}^\pm(k)$ are easily seen to have the direct sum decompositions

$$\begin{aligned} E(x,y;k) &= E^0(x,y;k) \dotplus E^\times(x,y;k), \\ \boldsymbol{V}(k) &= \boldsymbol{V}^0(k) \dotplus I_{n-1}, \\ \boldsymbol{V}_\#(k) &= \boldsymbol{V}_\#^0(k) \dotplus I_{n-1}, \end{aligned}$$

where

$$E^\times(x,y;k) = \begin{cases} +e^{ik(x-y)}I_{n-1}, & x>y, \operatorname{Im} k>0, \\ -e^{-ik(y-x)}I_{n-1}, & y>x, \operatorname{Im} k<0, \\ 0_{(n-1)\times(n-1)}, & \text{otherwise}, \end{cases}$$

$$\begin{aligned} &E^0(x,y;k) \\ &= \begin{cases} e^{i\lambda(x-y)}P^0_{p,-}(k) + e^{i\lambda(x+y-2x_0)}\boldsymbol{V}^0_\#(k)^{-1}P^0_{m,-}(k)P^0_{p,+}(k), & x>y>x_0, \\ -e^{i\lambda(y-x)}P^0_{p,+}(k) + e^{i\lambda(x+y-2x_0)}\boldsymbol{V}^0_\#(k)^{-1}P^0_{m,-}(k)P^0_{p,+}(k), & y>x>x_0, \\ -e^{i\lambda(y-x)}P^0_{m,+}(k) - e^{i\lambda(2x_0-x-y)}\boldsymbol{V}^0(k)^{-1}P^0_{p,+}(k)P^0_{m,-}(k), & x<y<x_0, \\ e^{i\lambda(x-y)}P^0_{m,-}(k) - e^{i\lambda(2x_0-x-y)}\boldsymbol{V}^0(k)^{-1}P^0_{p,+}(k)P^0_{m,-}(k), & y<x<x_0, \\ e^{i\lambda(x-y)}\boldsymbol{V}^0_\#(k)^{-1}P^0_{m,-}(k), & x>x_0>y, \\ -e^{i\lambda(y-x)}\boldsymbol{V}^0(k)^{-1}P^0_{p,+}(k), & y>x_0>x. \end{cases} \end{aligned}$$

for $k \in \mathbb{K}^+$, and $E^0(x,y;k)$ has the same form but with each $i\lambda$ replaced by $-i\lambda$ if $k \in \mathbb{K}^-$. Then the Green's function has the following properties:

$$E(x,z;k)E(z,y;k) = +E(x,y;k), \qquad x>z>y, \tag{2.10a}$$

$$E(x,z;k)E(z,y;k) = -E(x,y;k), \qquad x<z<y, \tag{2.10b}$$

$$E(x,x^-;k) - E(x,x^+;k) = I_{n+1}. \tag{2.10c}$$

Equation (2.10c) is easily verified by using that

$$\boldsymbol{V}^0_\#(k)^{-1}P^0_{m,-}(k) + \boldsymbol{V}^0(k)^{-1}P^0_{p,+}(k) = I_2.$$

2.4. Diagonalizing the free Hamiltonian. In this subsection we prove the absolute continuity of the free Hamiltonian H_0. We first define the (modified) Fourier transform $\mathbb{F}$ by

$$\hat{\psi}(\xi) = (\mathbb{F}\psi)(\xi) = \int_{-\infty}^{\infty} dx\, e^{i\xi(x-x_0)J}\psi(x), \tag{2.11a}$$

$$\psi(x) = (\mathbb{F}^{-1}\hat{\psi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi(x-x_0)J}\hat{\psi}(\xi). \tag{2.11b}$$

Theorem 2.2. *The free Hamiltonian $H_0 = iJ(d/dx) - V_f$ is an absolutely continuous selfadjoint operator on the direct sum of $n+1$ copies of $L^2(\mathbb{R})$ whose spectrum consists of two layers of $k \in (-\infty, -q_0] \cup [q_0, \infty)$ and, for $n \geq 2$, $n-1$ layers of $k \in \mathbb{R}$.*

P r o o f. Recall that $\boldsymbol{U}^\dagger H_0 \boldsymbol{U}$ is the orthogonal direct sum of a selfadjoint operator $[\boldsymbol{U}^\dagger H_0 \boldsymbol{U}]^{1+1}$ on $L^2(\mathbb{R})^{2\times 1}$ and $n-1$ copies of $-i(d/dx)$. Applying the Fourier transform (2.11) to $-i(d/dx)I_{n-1}$, we obtain as the resolvent an integral operator on $L^2(\mathbb{R})^{(n-1)\times 1}$ with integral kernel

$$\begin{aligned}\hat{E}^\times(\xi,\eta;k) &= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{i\xi(x-x_0)J}e^{-i\eta(y-x_0)J}E^\times(x,y;k)\\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\, \frac{e^{i(\xi-\eta)z}}{\eta - k} I_{n-1} = \frac{1}{\xi - k}\delta(\xi-\eta) I_{n-1},\end{aligned}$$

which implies the absolute continuity of $-i(d/dx)I_{n-1}$ with uniform spectral multiplicity $n-1$ on its spectrum $k \in \mathbb{R}$.

To prove the absolute continuity of H_0, it is sufficient to prove the absolute continuity of the operator $[\boldsymbol{U}^\dagger H_0 \boldsymbol{U}]^{1+1}$ whose resolvent is an integral operator on $L^2(\mathbb{R})^{2\times 1}$ with integral kernel

$$\hat{E}^0(\xi,\eta;k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{i\xi(x-x_0)J}e^{-i\eta(y-x_0)J}E^0(x,y;k). \tag{2.12}$$

This integral kernel has been evaluated in the Appendix as the sum of eight terms. For $\xi, \eta, z \in \mathbb{R}$, these terms contain $P^0_{p,\pm}(k)$, $P^0_{m,\pm}(k)$, $\boldsymbol{V}^0_\#(k)^{-1}$, $\boldsymbol{V}^0(k)^{-1}$, $(\lambda(k)I_2 \pm \xi J)^{-1}$, $(\lambda(k) \pm \eta J)^{-1}$, and $e^{i\lambda z}I_2$, all of which are 2×2 matrices $F(k)$ depending on $k \notin (-\infty, -q_0] \cup [q_0, \infty)$ having the property that $F(k) = F(k^*)^\dagger$. All of the entities $F(k+i\varepsilon)$ and $F(k-i\varepsilon)$ have finite limits as $\varepsilon \to 0^+$, provided $k < -q_0$ or $k > q_0$. As a result, the integral operator on $L^2(\mathbb{R})^{2\times 1}$ with kernel

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\left\{E^0(\xi,\eta;k-i\varepsilon) - E^0(\xi,\eta;k+i\varepsilon)\right\}$$

is bounded, provided $k < -q_0$ or $k > q_0$, which proves the absolute continuity of $[\boldsymbol{U}^\dagger H_0 \boldsymbol{U}]^{1+1}$. $\square$

3. Wave operators and scattering operator. In this section we prove that the wave operators exist and are asymptotically complete (in the sense of [9, 17]) and that the scattering operator is unitary.

Suppose $H_0 = iJ(d/dx) - V_f$ (with $\boldsymbol{q}_f(x) = \boldsymbol{q}_+$ for $x > x_0$ and $\boldsymbol{q}_f(x) = \boldsymbol{q}_-$ for $x < x_0$) is the free Hamiltonian and $H = iJ(d/dx)I_{n+1} - V$ (where $\boldsymbol{q}(x)$ satisfies (1.3)) is the full Hamiltonian. Since either operator is selfadjoint on the direct sum of $n+1$ copies of $L^2(\mathbb{R})$, then, according to Pearson's theorem [9, 17, 19, 20], it is sufficient to prove that the resolvent difference $(\zeta I - H)^{-1} - (\zeta I - H_0)^{-1}$ is trace-class in order to conclude that the wave operators $W_\pm$ and $Z_\pm$ defined by

$$W_\pm = \lim_{t\to\pm\infty} P_{\mathrm{ac}}(H) e^{itH} e^{-itH_0}, \qquad Z_\pm = \lim_{t\to\pm\infty} e^{itH_0} e^{-itH} P_{\mathrm{ac}}(H), \tag{3.1}$$

exist and are asymptotically complete in the sense that $W_\pm$ have the same closed range and $Z_\pm$ have the same closed range. Here $P_{\mathrm{ac}}(H)$ is the orthogonal projection onto the absolutely continuous subspace of H. In that case the *scattering operator*

$$S = Z_+ W_-$$

is unitary and commutes with the free Hamiltonian H_0. It is sufficient to prove that

$$\boldsymbol{U}^\dagger \left[(\zeta I - H)^{-1} - (\zeta I - H_0)^{-1} \right] \boldsymbol{U} \tag{3.2}$$

is trace-class. Put

$$\delta(x) = \begin{cases} \delta_p(x), & x > x_0, \\ \delta_m(x), & x < x_0. \end{cases}$$

Then (2.2) converts the matrix Zakharov-Shabat system into

$$\begin{aligned} \frac{\partial Y}{\partial x}(k,x) &= \begin{pmatrix} -ik & q(x) & 0_{1\times(n-1)} \\ q(x)^* & ik & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & ikI_{n-1} \end{pmatrix} Y(k,x) \\ &\quad + \begin{pmatrix} 0 & \delta(x) & \boldsymbol{q}_v(x) \\ \delta(x)^* & 0 & 0_{1\times(n-1)} \\ \boldsymbol{q}_v(x)^\dagger & 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix}, \end{aligned}$$

where $q(x) = q_+$ for $x > x_0$ and $q(x) = q_-$ for $x < x_0$. Hence,

$$\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U} = \begin{pmatrix} 0 & i\delta(x) & i\boldsymbol{q}_v(x) \\ -i\delta(x)^* & 0 & 0_{1\times(n-1)} \\ -i\boldsymbol{q}_v(x)^\dagger & 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & i\boldsymbol{\Delta}(x)^\dagger \\ -i\boldsymbol{\Delta}(x) & 0_{n\times n} \end{pmatrix},$$

where the row vector

$$\boldsymbol{\Delta}(x)^\dagger = \begin{pmatrix} \delta(x) & \boldsymbol{q}_v(x) \end{pmatrix} = \begin{pmatrix} \delta(x) & \boldsymbol{q}(x)\boldsymbol{v}_1^\dagger & \ldots & \boldsymbol{q}(x)\boldsymbol{v}_{n-1}^\dagger \end{pmatrix}$$

has its entries in $L^1(\mathbb{R})$. Consequently, we get the polar decomposition $\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U} = U(x)|\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|$, where

$$U(x) = \begin{pmatrix} 0 & iU^{\mathrm{dn}} \\ -iU^{\mathrm{up}} & 0_{n\times n} \end{pmatrix}, \ |\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}| = \begin{pmatrix} [\boldsymbol{\Delta}^\dagger\boldsymbol{\Delta}]^{1/2} & 0_{1\times n} \\ 0_{n\times 1} & [\boldsymbol{\Delta}\boldsymbol{\Delta}^\dagger]^{1/2} \end{pmatrix},$$

$\boldsymbol{\Delta}(x)^\dagger = U^{\mathrm{dn}}(x)[\boldsymbol{\Delta}(x)\boldsymbol{\Delta}(x)^\dagger]^{1/2}$, and $\boldsymbol{\Delta}(x) = U^{\mathrm{up}}(x)[\boldsymbol{\Delta}(x)^\dagger\boldsymbol{\Delta}(x)]^{1/2}$. Here we have not always written the dependence on x.

We first derive a preliminary result (cf. [5, 12]) on sufficient conditions for certain integral operators with kernel $K(x,y)$ to be Hilbert-Schmidt on $L^2(\mathbb{R})$. The proof proceeds by an easy estimation of $\int_{-\infty}^\infty dx \int_{-\infty}^\infty dy\,|K(x,y)|^2$ and is omitted.

Lemma 3.1. *Let $W, W_1, W_2 \in L^2(\mathbb{R})$. Then the following is true:*

1. *The integral operators with kernel $e^{i\zeta|x-y|}W(x)$ and $e^{i\zeta|x-y|}W(y)$ are Hilbert-Schmidt on $L^2(\mathbb{R})$ if $Im\,\zeta > 0$.*

2. *The integral operator with kernel $e^{i\zeta|x-y|}W_1(x)W_2(y)$ is Hilbert-Schmidt on $L^2(\mathbb{R})$ if $Im\,\zeta \geq 0$.*

3. *For $x_0 \in \mathbb{R}$, the integral operators with kernel $e^{i\zeta|x+y-2x_0|}W(x)$ and kernel $e^{i\zeta|x+y-2x_0|}W(y)$ are Hilbert-Schmidt on $L^2(\mathbb{R})$ if $Im\,\zeta > 0$.*

4. *For $x_0 \in \mathbb{R}$, the integral operator with kernel $e^{i\zeta|x+y-2x_0|}W_1(x)W_2(y)$ is Hilbert-Schmidt on $L^2(\mathbb{R})$ if $Im\,\zeta \geq 0$.*

We now prove the sufficient condition for applying Pearson's theorem.

Theorem 3.2. *Suppose $\boldsymbol{q}(x) - \boldsymbol{q}_f(x)$ is a row vector with entries in $L^2(\mathbb{R})$. Then, for $\zeta \notin \mathbb{R}$, the resolvent difference* (3.2) *is a trace-class operator.*

P r o o f. Mimicking the proof of [5, Prop. 2 and Thm. 4], we put

$$W(\zeta) = I + |\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}(\zeta I - H_0)^{-1}U|\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}.$$

Then, for $\zeta \notin \mathbb{R}$,

$$\boldsymbol{U}^\dagger\left[(\zeta I - H)^{-1} - (\zeta I - H_0)^{-1}\right]\boldsymbol{U} = -[\boldsymbol{U}^\dagger(\zeta I - H_0)^{-1}\boldsymbol{U}]U\times$$

$$\times\, |\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}W(\zeta)^{-1}|\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}[\boldsymbol{U}^\dagger(\zeta I - H_0)^{-1}\boldsymbol{U}],$$

where the invertibility of $W(\zeta)$ is clear from the selfadjointness of H. Thus it suffices to prove that, for $\zeta \in \mathbb{C}\setminus\mathbb{R}$, $|\boldsymbol{U}^\dagger(\zeta I - H_0)\boldsymbol{U}|^{1/2}U|\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}$ and $|\boldsymbol{U}^\dagger(H-H_0)\boldsymbol{U}|^{1/2}|\boldsymbol{U}^\dagger(\zeta I - H_0)\boldsymbol{U}|^{1/2}$ are Hilbert-Schmidt operators. This is immediate from parts 1 and 3 of Lemma 3.1 and the explicit expression for $E(x,y;k)$, since $\boldsymbol{\Delta}(x)$ is a column vector having its entries in $L^1(\mathbb{R})$. □

Applying Pearson's theorem [9, 17, 19, 20], we obtain the following result.

Theorem 3.3. *Suppose $\boldsymbol{q}(x) - \boldsymbol{q}_f(x)$ is a row vector with entries in $L^2(\mathbb{R})$. Then the wave operators $W_\pm$ and $Z_\pm$ defined by* (3.1) *exist and are asymptotically complete in the sense that $W_\pm$ have the absolutely continuous subspace of H as their range and $Z_\pm$ are onto. Moreover, the scattering operator*

$$S = Z_+W_-$$

is unitary.

The wave operators $W_\pm$ and $Z_\pm$ defined by (3.1) are partial isometries satisfying the following relations:

$$Z_\pm = (W_\pm)^\dagger, \tag{3.3a}$$

$$W_\pm[\mathcal{D}(H_0)] \subset \mathcal{D}(H) \text{ and } HW_\pm = W_\pm H_0, \tag{3.3b}$$

$$Z_\pm[\mathcal{D}(H)] \subset \mathcal{D}(H_0) \text{ and } Z_\pm H = H_0 Z_\pm, \tag{3.3c}$$

$$\operatorname{Im} W_\pm = [\operatorname{Ker} Z_\pm]^\perp = \operatorname{Im} P_{\rm ac}(H), \tag{3.3d}$$

$$\operatorname{Ker} W_\pm = [\operatorname{Im} Z_\pm]^\perp = \{0\}. \tag{3.3e}$$

Moreover, the absolutely continuous part of the full Hamiltonian H is unitarily equivalent to the free Hamiltonian H_0. Furthermore, the scattering operator and the free Hamiltonian commute:

$$SH_0 = (Z_+W_-)H_0 = Z_+(W_-H_0) = Z_+(HW_-)$$

$$= (Z_+H)W_- = (H_0Z_+)W_- = H_0(Z_+W_-) = H_0S.$$

A. Free hamiltonian resolvent kernel. Applying the Fourier transform $\mathbb{F}$ defined by (2.11) to the (transformed) free Hamiltonian $[\boldsymbol{U}^\dagger H_0 \boldsymbol{U}]^{1+1}$, we get a linear operator with resolvent kernel (2.12), where $J = \operatorname{diag}(1, -1)$ and $k \notin (-\infty, -q_0] \cup [q_0, \infty)$.

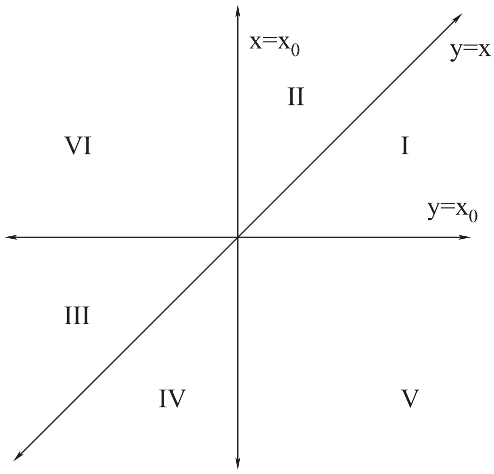

Fig. A.1. The six regions in which the euclidean plane is divided by the straight lines $x = y$, $x = x_0$, and $y = x_0$

To compute $\hat{E}^0(\xi, \eta; k)$ for $k \notin (-\infty, -q_0] \cup [q_0, \infty)$, we use the straight lines $x = x_0$, $y = x_0$, and $y = x$ to divide $\mathbb{R}^2$ into the six subregions corresponding to the subdomains used in defining $E^0(x, y; k)$. Distinguishing between the contributions I_1, II_1, III_1, and IV_1 of the first terms of $E^0(x, y; k)$ in the first four subdomains, the contributions V and VI of the last two subdomains, the sum $I_2 + II_2$ of the contributions of the second terms in the first two subdomains, and the sum $III_2 + IV_2$ of the contributions of the second terms in the last two subdomains, we get eight terms contributing to $\hat{E}^0(\xi, \eta; k)$, all of which are independent of $x_0 \in \mathbb{R}$. We get

$$I_1 = \frac{-1}{2\pi i} \int_0^\infty dz\, (\lambda I_2 + \eta J)^{-1} \left[e^{i(\xi - \eta) z J} - e^{i \xi z J} e^{i \lambda z} \right] P^0_{p,-}(k),$$

$$II_1 = \frac{1}{2\pi i} \int_0^\infty dz\, (\lambda I_2 - \xi J)^{-1} \left[e^{i(\xi - \eta) z J} - e^{-i \eta z J} e^{i \lambda z} \right] P^0_{p,+}(k),$$

$$III_1 = \frac{1}{2\pi i} \int_0^\infty dz\, (\lambda I_2 - \eta J)^{-1} \left[e^{-i(\xi - \eta) z J} - e^{-i \xi z J} e^{i \lambda z} \right] P^0_{m,+}(k),$$

$$IV_1 = \frac{-1}{2\pi i} \int_0^\infty dz\, (\lambda I_2 + \xi J)^{-1} \left[e^{-i(\xi-\eta)zJ} - e^{i\eta zJ} e^{i\lambda z} \right] P^0_{m,-}(k),$$

$$V = \frac{-1}{2\pi} (\lambda I_2 + \xi J)^{-1} (\lambda I_2 + \eta J)^{-1} \boldsymbol{V}^0_{\#}(k)^{-1} P^0_{m,-}(k),$$

$$VI = \frac{-1}{2\pi} (\lambda I_2 - \xi J)^{-1} (\lambda I_2 - \eta J)^{-1} \boldsymbol{V}^0(k)^{-1} P^0_{p,+}(k),$$

$$I_2 + II_2 = \frac{-1}{2\pi} (\lambda I_2 + \xi J)^{-1} (\lambda I_2 - \eta J)^{-1} \boldsymbol{V}^0_{\#}(k)^{-1} P^0_{m,-}(k) P^0_{p,+}(k),$$

$$III_2 + IV_2 = \frac{1}{2\pi} (\lambda I_2 - \xi J)^{-1} (\lambda I_2 + \eta J)^{-1} \boldsymbol{V}^0(k)^{-1} P^0_{p,+}(k) P^0_{m,-}(k).$$

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