A Kinetic Type Extended Model for Polarizable and Magnetizable Fluids.

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Abstract An elegant formulation of thermodynamics in electromagnetic fields has been provided by Liu and Müller and is based upon the conservation laws of mass, momentum and energy as well as on Maxwell's equations. However, in other physical context it has been shown the opportunity of considering an extended set of independent variables. Therefore, it is fitting to follow an extended approach also for charged fluids in electromagnetic fields; in literature this methodology has already been used, but only for the case of negligible effects of polarization and magnetization; here this restriction is removed and the general case treated. The entropy principle and the principle of material frame indifference are imposed; by using the methods of Extended Thermodynamics, we can see that they give very strong restrictions on the constitutive functions appearing in these balance laws.

1 Introduction.

In ordinary Thermodynamics, the conservation laws of mass (with density F), momentum (with density F_i) and energy (with density $\frac{1}{2}F_{ll}$) are used as field equations; in these equations, also the momentum flux density F_{ij} and the energy flux density $\frac{1}{2}F_{ill}$ occur, and they are linked to the independent variables F, F_i, $\frac{1}{2}F_{ll}$ and to their gradients through the state equations and the Navier-Stokes and Fourier laws. But in this way parabolic equations are obtained which yield infinite speeds of shocks propagation. In extended thermodynamics (see [1] and subsequent papers summarized in [2]) the aim has been realized to obtain an iperbolic set of field equations (and symmetric too) in the following way

- Consider as independent variables F, F_i, F_{ij}, F_{ill} (in other words, also the above fluxes have been included); for this increased number of independent variables, consider also a corresponding increased number of field equations.
- Link the new fluxes, which appear in these equations, only to the independent variables and not to their gradients. Restrict the generality of these links, or constitutive equations, by imposing the principle of entropy and that of Galilean invariance.

In this way a symmetric hyperbolic set of field equations are obtained, consequently yielding finite speeds of shocks propagation and continuous dependence on the initial conditions; therefore, they are more physically significant than those of ordinary Thermodynamics. This last one can also be recovered from those of Extended Thermodynamics as first approximation of a particular iterative procedure.
However, in [1], the flux appearing in a field equation is the independent variable of the subsequent equation; it follows that the original model describes only mono-atomic gases. We have verified that, also from the mathematical point of view, this structure leads to results which are too much restrictive for polarizable and magnetizable fluids; for example, also at equilibrium we obtain polarization effects without magnetization, which fact is physically unacceptable. This shows that the theory knows how polarization and magnetization cannot occur in mono-atomic gases! The reason is that in this case the model doesn't take into account the interactions between atoms and molecules.
In [3] it is shown how, also in Extended Thermodynamics, field equations can be considered which overcome this problem, and the fluxes are called F_k, G_{ik}, G_{ijk}, G_{ikll} (the first of these is still the momentum density, obviously); however, in [3] such constitutive functions have not been found by imposing the principles of entropy and that of Galilean invariance. This result has been recently achieved by some of us in [4] with a method akin to that of the kinetic theory, so that it has been called "A kinetic type extended model ...".
Here we want furtherly improve the model so that it may well describe also polarizable and magnetizable fluids. To this end we have in literature only models in the framework of ordinary Thermodynamics, such us [5]; here we want to obtain a model in the framework of Extended Thermodynamics, because it leads to more physically significant results

as seen above. We consider the following extended set of field equations. The first four of these have been found by applying the general guidelines of ref. [3]; note the contribution of the Lorentz force in the right-hand sides, and that of a term (in third and fourth equation) which takes into account external supplies other than body forces, according to the note on page 129 of ref. [3]. The eqs. $(1)_{1,2}$, (2), (3) and the trace of eq. $(1)_3$ are those studied by Liu and Müller [5] in the non extended approach.
The subsequent four equations are the Maxwell equations with electric field E_i and density of magnetic flux B_i, while the last two are definitions of the current J_i and of the charge density q in terms of the Polarization P_k, Magnetization M_j, the free current j_i^F and the free charge density q^F.

$$\partial_t F + \partial_k F_k = 0 \quad , \quad \partial_t F_i + \partial_k G_{ik} = qE_i + \epsilon_{iqp} J_q B_p \, , \tag{1}$$

$$\partial_t F_{ij} + \partial_k G_{ijk} = \frac{2}{F} F_{(i}(qE_{j)} + \epsilon_{j)qp} J_q B_p) + $$
$$+ \frac{2}{3}(E_r + \epsilon_{rqp} v_q B_p)(J_r - qv_r)\delta_{ij} + P_{<ij>} \, ,$$

$$\partial_t F_{ill} + \partial_k G_{ikll} = \frac{3}{F} F_{(il}(qE_{l)} + \epsilon_{l)qp} J_q B_p) +$$
$$+ \frac{10}{3F}(E_r + \epsilon_{rqp} v_q B_p)(J_r - qv_r) F_i + P_{ill} \, ,$$

$$\partial_t B_i + \epsilon_{ijk}\partial_j E_k = 0 \quad , \quad -\mu_0\epsilon_0 \partial_t E_i + \epsilon_{ijk}\partial_j B_k = \mu_0 J_i \, , \tag{2}$$
$$\partial_k B_k = 0 \, , \; \epsilon_0 \partial_k E_k = q \, ,$$

$$\partial_t P_i + \partial_k(\epsilon_{ijk} M_j + 2P_{[i} v_{k]}) = J_i - J_i^F \quad , \quad \partial_k P_k = q^F - q \, , \tag{3}$$

where $v_k = \frac{F_k}{F}$ is the velocity, ϵ_{ijk} is the Levi-Civita symbol, μ_0 the vacuum permeability, ϵ_0 the dielectric constant. The conservation of charge $\partial_t(q - q^F) + \partial_k(J_k - J_k^F) = 0$ is a consequence of (3). We note that another possible approach is to consider (3) not as field equations, but as definitions of J_i and q; the remaining eqs. (1), (2) are still a system of first order partial differential equations, even if the time and space derivatives occur also in the right-hand sides, through J_i and q. But in this way the divergence form is lost; for this reason we have chosen a different approach. We stress, once again, that in this set of equations the independent variables are F, F_i, F_{ij}, F_{ill}, B_i, E_i and P_i; but also the quantities G_{ik}, G_{ijk}, G_{ikll}, $P_{<ij>}$, P_{ill}, M_i, $J_i^* = J_i^F - q^F v_i$ occur in this system, so that they are unknown quantities for which closure relations are needed. The main result of this work are the expressions of these constitutive functions. They can be found by eliminating the parameters λ, λ_i, λ_{ij}, λ_{ill}, β_i, ϵ_i, π_i between the subsequent eqs. (6), $(7)_{2-4}$, $(8)_3$ which are expressed in terms of the functions h' and ϕ'_k, whose expressions are reported in the subsequent eqs. (16) and (17).
The arguments which allows us to find them are usual in Extended Thermodynamics, i.e., to impose that every solution of our system (1), (2), (3) satisfies a supplementary conservation law $\partial_t h + \partial_k \phi_k = \sigma \geq 0$. This amounts in assuming the existence of Lagrange multipliers λ, λ_i, λ_{ij}, λ_{ill}, β_i, ϵ_i, π_i, b, ϵ, π such that

$$dh = \lambda dF + \lambda_i dF_i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill} + \beta_i dB_i + \epsilon_i dE_i(-\mu_0\epsilon_0) + \pi_i dP_i \, ,$$
$$d\phi_k = \lambda dF_k + \lambda_i dG_{ik} + \lambda_{ij} dG_{ijk} + \lambda_{ill} dG_{illk} + \beta_i \epsilon_{ikj} dE_j + \tag{4}$$
$$+ \epsilon_i \epsilon_{ikj} dB_j + \pi_i d(\epsilon_{ikj} M_j + 2P_{[i} v_{k]}) + b dB_k + \epsilon\epsilon_0 dE_k + \pi dP_k \, ,$$

besides a residual inequality which we leave out for the sake of brevity.
By taking λ, λ_i, λ_{ij}, λ_{ill}, β_i, ϵ_i, π_i as independent variables, and by defining

$$h' = \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ill} F^{ill} + \beta_i B_i - \mu_0\epsilon_0 \epsilon_i E_i + \pi_i P_i - h \, , \tag{5}$$
$$\phi'_k = \lambda F_k + \lambda_i G_{ik} + \lambda_{ij} G_{ijk} + \lambda_{ill} G_{illk}$$
$$+ \beta_i \epsilon_{ikj} E_j + \epsilon_i \epsilon_{ikj} B_j + \pi_i(\epsilon_{ikj} M_j + 2P_{[i} v_{k]}) - \phi_k \, , \text{ they become}$$

$$F = \frac{\partial h'}{\partial \lambda}, \quad F^i = \frac{\partial h'}{\partial \lambda_i}, \quad F^{ij} = \frac{\partial h'}{\partial \lambda_{ij}}, \quad F^{ill} = \frac{\partial h'}{\partial \lambda_{ill}} \, , \tag{6}$$
$$B_i = \frac{\partial h'}{\partial \beta_i}, \quad -\mu_0\epsilon_0 E_i = \frac{\partial h'}{\partial \epsilon_i}, \quad P_i = \frac{\partial h'}{\partial \pi_i} ;$$

$$\frac{\partial \phi'_k}{\partial \lambda} = \frac{\partial h'}{\partial \lambda^k} - b\frac{\partial^2 h'}{\partial \lambda \partial \beta_k} + \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \lambda \partial \epsilon_k} - \pi \frac{\partial^2 h'}{\partial \lambda \partial \pi_k} \, , \tag{7}$$
$$G_{ik} = \frac{\partial \phi'_k}{\partial \lambda_i} + b\frac{\partial^2 h'}{\partial \lambda_i \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \lambda_i \partial \epsilon_k} + \pi \frac{\partial^2 h'}{\partial \lambda_i \partial \pi_k} \, ;$$
$$G_{ijk} = \frac{\partial \phi'_k}{\partial \lambda_{ij}} + b\frac{\partial^2 h'}{\partial \lambda_{ij} \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \lambda_{ij} \partial \epsilon_k} + \pi \frac{\partial^2 h'}{\partial \lambda_{ij} \partial \pi_k} \, ,$$
$$G_{illk} = \frac{\partial \phi'_k}{\partial \lambda_{ill}} + b\frac{\partial^2 h'}{\partial \lambda_{ill} \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \lambda_{ill} \partial \epsilon_k} + \pi \frac{\partial^2 h'}{\partial \lambda_{ill} \partial \pi_k} \, ,$$

$$\frac{\epsilon_{ikj}}{-\mu_0\epsilon_0}\frac{\partial h'}{\partial \epsilon_j} = \frac{\partial \phi'_k}{\partial \beta_i} + b\frac{\partial^2 h'}{\partial \beta_i \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \beta_i \partial \epsilon_k} + \pi\frac{\partial^2 h'}{\partial \beta_i \partial \pi_k}, \tag{8}$$
$$\epsilon_{ikj}\frac{\partial h'}{\partial \beta_j} = \frac{\partial \phi'_k}{\partial \epsilon_i} + b\frac{\partial^2 h'}{\partial \epsilon_i \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \epsilon_i \partial \epsilon_k} + \pi\frac{\partial^2 h'}{\partial \epsilon_i \partial \pi_k},$$
$$\epsilon_{ikj}M_j + 2P_{[i}v_{k]} = \frac{\partial \phi'_k}{\partial \pi_i} + b\frac{\partial^2 h'}{\partial \pi_i \partial \beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \pi_i \partial \epsilon_k} + \pi\frac{\partial^2 h'}{\partial \pi_i \partial \pi_k},$$
$$0 = \frac{\partial \phi'_{(k}}{\partial \pi_{i)}} + b\frac{\partial^2 h'}{\partial \pi_{(i}\partial \beta_{k)}} - \frac{\epsilon}{\mu_0}\frac{\partial^2 h'}{\partial \pi_{(i}\partial \epsilon_{k)}} + \pi\frac{\partial^2 h'}{\partial \pi_i \partial \pi_k}.$$

The eq. $(8)_4$ is the symmetric part of $(8)_3$, after that $(8)_3$ remains simply the definition of magnetization M_i. We note that, by dropping eqs. $(6)_{5,6,7}$, (8) and calculating the remaining ones in $\beta_i = 0$, $\epsilon_i = 0$, $\pi_i = 0$, $b = 0$, $\epsilon = 0$, $\pi = 0$, we obtain an important subsystem, i.e., the equation of the extended approach to dense gases and macromolecular fluids. These have been studied in [4] and we can use here the results. Similarly, by dropping eqs, $(6)_{1-4,7}$, (7), $(8)_{3,4}$ and calculating the remaining ones in $\lambda = 0$, $\lambda_i = 0$, $\lambda_{ij} = 0$, $\lambda_{ill} = 0$, $\pi = 0$, $\pi_i = 0$, we obtain the Maxwell equations. In the next section we will exploit their implications to eqs. (6), (7) and (8). At last, in section 3, we will consider the general case.

2 A supplementary conservation law for Maxwell equations

We have to consider the eq. $(6)_{5,6}$ and $(8)_{1,2}$ with $\pi = 0$ i.e.

$$B_i = \frac{\partial h'}{\partial \beta_i}; \quad -\mu_0\epsilon_0 E_i = \frac{\partial h'}{\partial \epsilon_i}, \tag{9}$$
$$\frac{1}{-\mu_0\epsilon_0}\epsilon_{ikj}\frac{\partial h'}{\partial \epsilon_i} = \frac{\partial \phi'_k}{\partial \beta_i} + b\frac{\partial^2 h'}{\partial \beta_i \partial \beta_k} - \frac{\epsilon_0}{\mu_0}\frac{\partial^2 h'}{\partial \beta_i \partial \epsilon_k},$$
$$\epsilon_{ikj}\frac{\partial h'}{\partial \beta_j} = \frac{\partial \phi'_k}{\partial \epsilon_i} + b\frac{\partial^2 h'}{\partial \epsilon_i \partial \beta_k} - \frac{\epsilon_0}{\mu_0}\frac{\partial^2 h'}{\partial \epsilon_i \partial \epsilon_k};$$

clearly, here we haven' t to impose the Galilean invariance principle, with decomposition in velocity dependent and independent parts; in fact, the velocity doesn't occur in this equations. For this reason we have assumed a supplementary conservation law and not an entropy principle. From the representation theorems [6], [7] and [8] we know that $\phi'_k = \varphi_1\epsilon_k + \varphi_2\beta_k + \varphi_3\epsilon_{krs}\beta_r\epsilon_s$ with φ_1, φ_2, φ_3, h', b and ϵ functions of $G_{11} = \epsilon_i\epsilon_i$, $G_{12} = \epsilon_i\beta_i$, $G_{22} = \beta_i\beta_i$. After that the symmetric parts with respect to i and k of $(9)_{3,4}$ give 2 linear combinations of $\epsilon_i\epsilon_k$, $\epsilon_{(i}\beta_{k)}$, $\beta_i\beta_k$, δ_{ik}, $\epsilon_{(i}\epsilon_{k)rs}\epsilon_r\beta_s$ and $\beta_{(i}\epsilon_{k)rs}\epsilon_r\beta_s$ which must be zero; by setting equal to zero the coefficients of the last 2 of the above tensors, we find that φ_3 is a constant. The skew-symmetric parts, with respect to i and k, of eqs. $(9)_{3,4}$ are linear combinations of $\epsilon_{[i}\beta_{k]}$, $\epsilon_{ikj}\epsilon_j$, $\epsilon_{ikj}\beta_j$; putting equal to zero the coefficients of these last 2 tensors, we find

$$\frac{\partial h'}{\partial G_{11}} = \frac{\mu_0\epsilon_0}{2}\varphi_3; \quad \frac{\partial h'}{\partial G_{12}} = 0; \quad \frac{\partial h'}{G_{22}} = \frac{1}{2}\varphi_3 \quad \text{i.e.},$$
$$h' = \frac{1}{2}\varphi_3(G_{22} + \mu_0\epsilon_0 G_{11}) + \text{const} = \frac{1}{2}\varphi_3(\beta_i\beta_i + \mu_0\epsilon_0\epsilon_i\epsilon_i) + \text{const}$$

After that, what remains of eq. $(9)_3$ shows that

$$\frac{\partial \varphi_1}{\partial G_{12}} = 0, \quad \frac{\partial \varphi_1}{\partial G_{22}} = 0, \quad \frac{\partial \varphi_2}{\partial G_{12}} = 0, \quad \frac{\partial \varphi_2}{\partial G_{22}} = 0, \quad \varphi_2 = -b\varphi_3$$

and what remains of $(11)_4$ gives $\quad \dfrac{\partial \varphi_1}{\partial G_{11}} = 0, \quad \dfrac{\partial \varphi_2}{\partial G_{11}} = 0; \quad \varphi_1 = \epsilon\epsilon_0\varphi_3;$

in other words ϵ, b, φ_1, φ_2 and φ_3 are constant and $\varphi_1 = \epsilon\epsilon_0\varphi_3$, $\varphi_2 = -b\varphi_3$.

3 The case with polarization and magnetization

Consider now the general case, the problem of finding the functions

$$h'(\lambda, \lambda_i, \lambda_{ij}, \lambda_{ill}, \beta_i, \epsilon_i, \pi_i) \quad \text{and} \quad \phi'_k(\lambda, \lambda_i, \lambda_{ij}, \lambda_{ill}, \beta_i, \epsilon_i, \pi_i) \quad \text{satisfying}$$

eqs. (6), (7) and (8). We have already determined, in ref. [4] , their expressions in $\beta_i = 0$, $\epsilon_i = 0$, $\pi_i = 0$. Let us define now the functions $\Delta h'$ and $\Delta\phi'_k$ from

$$h'(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},\beta_i,\epsilon_i,\pi_i) = h'(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},0,0,0) + \Delta h' \qquad (10)$$
$$\phi'_k(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},\beta_i,\epsilon_i,\pi_i) = \phi'_k(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},0,0,0) + \Delta\phi'_k\,,$$

and note that they become zero when calculated in $\beta_i = 0$, $\epsilon_i = 0$, $\pi_i = 0$. Substitute eqs. (10) in the conditions emerging from (6), (7) and (8), i.e., $(7)_1$, $(8)_{1,2,4}$ thus obtaining

$$\frac{\partial\Delta\phi'_k}{\partial\lambda} = \frac{\partial\Delta h'}{\partial\lambda^k} - b\frac{\partial^2\Delta h'}{\partial\lambda\partial\beta_k} + \frac{\epsilon}{\mu_0}\frac{\partial^2\Delta h'}{\partial\lambda\partial\epsilon_k} - \pi\frac{\partial^2\Delta h'}{\partial\lambda\partial\pi_k}\,, \qquad (11)$$
$$\frac{\epsilon_{ikj}}{-\mu_0\epsilon_0}\frac{\partial\Delta h'}{\partial\epsilon_j} = \frac{\partial\Delta\phi'_k}{\partial\beta_i} + b\frac{\partial^2\Delta h'}{\partial\beta_i\partial\beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2\Delta h'}{\partial\beta_i\partial\epsilon_k} + \pi\frac{\partial^2\Delta h'}{\partial\beta_i\partial\pi_k}\,,$$
$$\epsilon_{ikj}\frac{\partial\Delta h'}{\partial\beta_j} = \frac{\partial\Delta\phi'_k}{\partial\epsilon_i} + b\frac{\partial^2\Delta h'}{\partial\epsilon_i\partial\beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial^2\Delta h'}{\partial\epsilon_i\partial\epsilon_k} + \pi\frac{\partial^2\Delta h'}{\partial\epsilon_i\partial\pi_k}\,,$$
$$0 = \frac{\partial\Delta\phi'_{(k}}{\partial\pi_{i)}} + b\frac{\partial^2\Delta h'}{\partial\pi_{(i}\partial\beta_{k)}} - \frac{\epsilon}{\mu_0}\frac{\partial^2\Delta h'}{\partial\pi_{(i}\partial\epsilon_{k)}} + \pi\frac{\partial^2\Delta h'}{\partial\pi_i\partial\pi_k}\,.$$

When considering only the Maxwell equations, we have obtained that b and ϵ are constants; this suggests to restrict ourselves, also in this general case, to the solutions with b, ϵ and π not depending on λ, β_i, ϵ_i, π_i, for the sake of simplicity. By defining

$$\phi''_k = \Delta\phi'_k + b\frac{\partial\Delta h'}{\partial\beta_k} - \frac{\epsilon}{\mu_0}\frac{\partial\Delta h'}{\partial\epsilon_k} + \pi\frac{\partial\Delta h'}{\partial\pi_k} \qquad (12)$$

the eqs. (11) become

$$\frac{\partial\Delta h'}{\partial\lambda^k} = \frac{\partial\phi''_k}{\partial\lambda};\quad \frac{-\epsilon_{ikj}}{\mu_0\epsilon_0}\frac{\partial\Delta h'}{\partial\epsilon_j} = \frac{\partial\phi''_k}{\partial\beta_i}; \qquad (13)$$
$$\epsilon_{ikj}\frac{\partial\Delta h'}{\partial\beta_j} = \frac{\partial\phi''_k}{\partial\epsilon_i};\quad 0 = \frac{\partial\phi''_{(k}}{\partial\pi_{i)}}\,.$$

The symmetric parts with respect to i and k of $(13)_{2-4}$ show (with the same proof which deduces a rigid motion if the deformation tensor is zero) that ϕ''_k is linear both in ϵ_i that in β_i and in π_i ,i.e.,

$$\phi''_k = \phi_{kabc}\beta_a\epsilon_b\pi_c + \phi^3_{kab}\beta_a\epsilon_b + \phi^2_{kab}\pi_a\epsilon_b + \phi^1_{kab}\pi_a\beta_b +, \qquad (14)$$
$$+\ \phi^1_{ka}\epsilon_a + \phi^2_{ka}\beta_a + \phi^3_{ka}\pi_a + \phi''^0_k$$

where ϕ_{kabc}, ϕ^i_{kab}, ϕ^i_{ka}, ϕ''^0_k doesn't depend on β_i, ϵ_i and π_i; moreover, still the symmetric parts of $(13)_{2-4}$ show that ϕ_{kabc}, ϕ^i_{kab}, ϕ^i_{ka} change sign when we exchange the index k with whatever of the other indices. But we can exchange whatever couple of indices trough 3 changes of indices involving the first one; it follows that ϕ_{kabc}, ϕ^i_{kab}, ϕ^i_{ka} are skew-symmetric tensors for every couple of indices. But in ϕ_{kabc} at least one of the indices 1 2 3 occurs 2 times; therefore, we have $\phi_{kabc} = 0$. Moreover, ϕ^i_{kab} is not zero only when $k\ a\ b$ is 1 2 3 or anyone of its permutations; therefore, it is proportional to ϵ_{kab}. In other words, the scalars $\varphi^i(\lambda,\lambda_r,\lambda_{rs},\lambda_{rll})$ and the vectors $v^i_b(\lambda,\lambda_r,\lambda_{rs},\lambda_{rll})$ exist, such that

$$\phi^i_{kab} = \varphi^i\epsilon_{kab};\quad \phi^i_{ka} = \epsilon_{kab}v^i_b\,. \qquad (15)$$

These partial results simplify very much the exploitation of conditions (13), although the passages remain long and tedious, so that we report simply the final results, i.e., the expressions for the functions $\Delta h'$ and $\Delta\phi'_k$; they are the first three rows of the following eq. (16) and the first five rows of the eq. (17), respectively. The remaining rows are the expressions of $h'(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},0,0,0)$ and $\phi'_k(\lambda,\lambda_i,\lambda_{ij},\lambda_{ill},0,0,0)$ found in ref. [4] (up to second order in the variables λ_i, $\lambda_{<ij>}$, λ_{ill}); their sum, according to eq. (10), gives the functions h' and ϕ'_k, i.e.,

$$h' = \frac{1}{2}\varphi^3(\mu_0\epsilon_0\epsilon_j\epsilon_j + \beta_j\beta_j) - \mu_0\epsilon_0\varphi^1\epsilon_j\pi_j + \varphi^2\beta_j\pi_j + \epsilon_{rjb}\lambda_r\epsilon_j v^{11}_b + \qquad (16)$$
$$+\epsilon_{rjb}\lambda_r\beta_j v^{21}_b + \mu_0\epsilon_0\epsilon_j(v^{20}_j + v^{21}_j\lambda) - \beta_j(v^{10}_j + v^{11}_j\lambda) +$$
$$+\frac{\partial v^3_{b_0}}{\partial\lambda}\epsilon_{kab}\lambda_k\pi_a + \pi_r H_r(\lambda,\lambda_{ia},\lambda_{ill},\pi_k) +$$
$$-\frac{8}{27\cdot 35}G'(\lambda)\lambda_{ll}^{-3/2} - \frac{2}{21}G'(\lambda)\lambda_{ll}^{-7/2}\lambda^i\lambda_{ill} +$$
$$\frac{2}{9\cdot 35}G''(\lambda)\lambda_{ll}^{-5/2}\lambda_i\lambda^i - \frac{2}{105}G'(\lambda)\lambda_{ll}^{-7/2}\lambda_{<ij>}\lambda_{<ij>} +$$
$$\frac{1}{2}G(\lambda)\lambda_{ll}^{-9/2}\lambda_{ill}\lambda_{ill}\,,$$

$$\begin{aligned}\phi_k' = {} & \varphi^3(\epsilon_{kab}\beta_a\epsilon_b - b\beta_k + \epsilon\epsilon_0\epsilon_k) + \varphi^2(\epsilon_{kab}\pi_a\epsilon_b - b\pi_k - \pi\beta_k) + \\ & + \varphi^1(\epsilon_{kab}\pi_a\beta_b - \epsilon\epsilon_0\pi_k + \epsilon_0\mu_0\pi\epsilon_k) + v_b^{10}\epsilon_{kab}\epsilon_a + v_b^{20}\epsilon_{kab}\beta_a + \\ & + v_b^{21}(\epsilon_{kab}\beta_a\lambda + \epsilon_b\lambda_k - \delta_{kb}\epsilon_r\lambda_r) + \\ & + v_b^{11}\left(\epsilon_{kab}\epsilon_a\lambda - \beta_b\lambda_k\frac{1}{\epsilon_0\mu_0} + \delta_{kb}\beta_r\lambda_r\frac{1}{\epsilon_0\mu_0}\right) + \\ & - \pi\left[\pi_r\frac{\partial H_r}{\partial \pi_k} + H_k - (H_k)_{\pi_r=0}\right] + \epsilon_{kab}\pi_a[v_{b1}^3 + v_{b0}^3] + \\ & \frac{4}{9\cdot 35}G'(\lambda)\lambda_{ll}^{-5/2}\lambda_k + \left[-\frac{2}{21}G(\lambda)\lambda_{ll}^{-7/2} + f_1(\lambda_{ll})\right]\lambda_{kll} + \\ & \left[\frac{2}{5}G(\lambda)\lambda_{ll}^{-3/2} + f_2(\lambda_{ll})\right]\lambda_{<kr>}\lambda_{rll} - \frac{4}{105}G'(\lambda)\lambda_{ll}^{-7/2}\lambda_{<kr>}\lambda_r .\end{aligned} \tag{17}$$

Here, φ^i, v_b^{11}, v_b^{21}, v_b^{10}, v_b^{20} are functions of λ_{rs}, λ_{rll},
b, ϵ, π, v_{b1}^3 are functions of λ_r, λ_{rs}, λ_{rll},
v_{b0}^3 is function of λ, λ_{rs}, λ_{rll},
H_r is function of λ, λ_{rs}, λ_{rll}, π_r ;
$G(\lambda)$ is function of λ, $f_1(\lambda_{ll})$ and $f_2(\lambda_{ll})$ are functions of λ_{ll};
they are arbitrary functions restricted only by

$$\pi\frac{\partial^2 v_{b0}^3}{\partial\lambda^2} = 0 \text{ and } \pi\left(H_k\right)_{\pi_r=0} = (bv_k^{11} + \epsilon\epsilon_0 v_k^{21})\lambda + \pi H_k^*(\lambda_{ia}, \lambda_{ill}), \tag{18}$$

with H_k^* another arbitrary function of its variables.
The first terms of eq. (16) and the first one of eq. (17) are the same of the corresponding ones in sect 2, for the Maxwell equations. The only difference is that here φ^3 may depend on λ_{ij}, λ_{ill} , while in sect 2 it was a constant. It is easy to verify that, in this way, eqs. (11) are satisfied.
The expressions (16) and (17) can now be inserted in eqs. (6), $(7)_{2-4}$, $(8)_3$ and give F, F_i, F_{ij}, F_{ill}, B_i, E_i, P_i, G_{ik}, G_{ijk}, G_{ikll}, M_i as functions of the parameters λ, λ_i, λ_{ij}, λ_{ill}, β_i, ϵ_i, π_i. The first ones of these functions can be used to obtain the parameters as functions of F, F_i, F_{ij}, F_{ill}, B_i, E_i, P_i; by inserting these in the remaining ones, we obtain the constitutive functions G_{ik}, G_{ijk}, G_{ikll}, M_i as functions of the independent variables F, F_i, F_{ij}, F_{ill}, B_i, E_i, P_i. In this way the requested closure has been obtained. We apologize because we cannot report these passages in the only 8 pages allowed for these proceedings; the interested reader may do them by himself, because they are straightforward, or may ask us to send them privately. The same thing we have to say for the other constitutive functions $P_{<ij>}$, P_{ill}, $J_i^* = J_i^F - q^F v_i$.

Conclusions
We retain the results of the present paper very satisfactory, because they allow to study also polarizable and magnetizable fluids in the framework of the well established theory of Extended Thermodynamics. The field equations to be solved are (1), (2) and (3) closed in the above mentioned way; although apparently complicate they can be put in the symmetric hyperbolic form by simply changing the independent variables, so predicting finite speeds of wave propagations. There remains to understand the physical meaning of the arbitrary functions still remaining in our closure. Some of them depend upon the particular fluid treated, and are related to the state functions; and the others? are zero, perhaps? This will be argument of further investigation.

Thanks: We thank anonymous referees; they helped us in improving the presentation of this article.

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