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Exact solutions of the Hirota equation and vortex filaments motion

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HIGHLIGHTS

- We construct an explicit soliton solution formula for the Hirota equation.
- We get our solutions via the Inverse Scattering Method and the matrix triplet method.
- Our formula contains a new class of solutions called multipole soliton solutions.
- Every solution corresponds to a vortex filament motion with nonzero axial velocity.

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ABSTRACT

By using the Inverse Scattering Transform we construct an explicit soliton solution formula for the Hirota equation. The formula obtained allows one to get, as a particular case, the N-soliton solution, the breather solution and, most relevantly, a new class of solutions called multipole soliton solutions. We use these exact solutions to study the motion of a vortex filament in an incompressible Euler fluid with nonzero axial velocity.

1. Introduction

In 1973 Hirota [1] considered the following equation:

$$iq_t + 3i\alpha|q|^2q_x + \rho q_{xx} + i\sigma q_{xxx} + \delta|q|^2q = 0, \tag{1.1}$$

where subscripts denote partial derivatives, q is a scalar function, $(x, t) \in \mathbb{R}^2$, i is the imaginary unit, and $\alpha, \rho, \sigma, \delta$ are real constants which satisfy $\alpha\rho = \sigma\delta$. In his paper [1] Hirota, applying the method which takes his name [2], obtained the N-soliton solutions for this equation. Eq. (1.1) can be written as

$$iq_t - \alpha_2\left[q_{xx} + 2|q|^2q\right] + i\alpha_3\left[q_{xxx} + 6|q|^2q_x\right] = 0, \tag{1.2}$$

where we have chosen $\alpha = 2\alpha_3$, $\delta = -2\alpha_2$, $\rho = -\alpha_2$ and $\sigma = \alpha_3$ in such a way that the constraint $\alpha\rho = \sigma\delta$ is satisfied. We observe that for $\alpha_2 = -1,\ \alpha_3 = 0$ we get the focusing Nonlinear Schrödinger (NLS) equation and for $\alpha_2 = 0,\ \alpha_3 = 1$ Eq. (1.2) reduces to the complex modified Korteweg–de Vries (cmKdV) equation. Eq. (1.2) is integrable because it is the sum of the commuting integrable flows given by the NLS and cmKdV which are PDEs belonging to the same hierarchy. In 1991 Fukumoto and Miyazaki [3] showed the relevance of the Hirota equation (1.2) in the modeling of the vortex string motion for a three dimensional Euler incompressible fluid. A complete analytic study of this problem is at the moment technically impossible and some approximation is necessary. The classical one is the local induction

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approximation (LIA) first developed in [4,5] for a model with zero axial velocity. In this case it is assumed that the main contribution to the self-interaction of the vortex string in a point is given by a finite length of the string about the point. The equation of motion for the vortex filament motion is the well-known purely binormal velocity motion

$$\vec{X}_t = \alpha_2 \vec{X}_s \times \vec{X}_{ss} \tag{1.3}$$

where $\vec{X} = \vec{X}(s,t)$ is the position at time t of the vortex line parameterized by the arclength s, α_2 is a parameter depending on the vortex circulation (see Eq. (1.1) of [3]) and $\times$ denotes the vector product.

The most relevant result of [3] is the extension of the LIA approximation to the system

$$\vec{X}_t = \alpha_2 \vec{X}_s \times \vec{X}_{ss} + \alpha_3 \left(\vec{X}_{sss} + \frac{3}{2} \vec{X}_{ss} \times (\vec{X}_s \times \vec{X}_{ss}) \right) \tag{1.4}$$

where the new term proportional to α_3 encodes the correction to the LIA due to a nonzero constant axial velocity along the vortex filament (see Eq. (3.2) of [3]). The main approximation present in this system are the elimination of the nonlocal self-induction of the vortex and the possible variation of the vortex core along the vortex string (see [3] beginning of Section 3).

In the case of constant vorticity and null velocity inside the vortex core the resulting model is equivalent to the NLS equation, by means of the Hasimoto map $\psi = 2\kappa \exp\left(i \int^s \tau(s',t) ds'\right)$ [6] where κ is the curvature and τ is the torsion of the filament $\vec{X}$ whose evolution follows (1.3). The extension to the LIA present in [3] gives rise to a 2-fields PDE system equivalent to (1.2) by means of the same Hasimoto map obtaining

$$\begin{aligned} &\kappa_t + \alpha_2(2\kappa_s\tau + 2\kappa\tau_s) + \alpha_3\left(6\kappa^2\kappa_s + \kappa_{sss} - 3\kappa_s\tau^2 - 3\kappa\tau\tau_s\right) = 0, \\ &\tau_t - \alpha_2\left(\left(\frac{\kappa_{ss}}{\kappa}\right)_s - 2\tau\tau_s + 4\kappa\kappa_s\right) - \alpha_3\left(12\kappa\kappa_s\tau + 6\kappa^2\tau_s + 3\left(\frac{\kappa_{ss}\tau}{\kappa}\right)_s + 3\left(\frac{\kappa_s\tau_s}{\kappa}\right)_s + \tau_{sss} - 3\tau^2\tau_s\right) = 0. \end{aligned} \tag{1.5}$$

where now κ and τ are respectively curvature and torsion of a curve $\vec{X}$ whose evolution is given by (1.4).

The first topic of this paper is to find explicit solutions for Eq. (1.2) in order to allow a straight evaluation of the contribution of axial velocity in the vortex string motion. Even though the paper is devoted to the explicit study of solutions without gradient catastrophes, we hope that the method developed here can be extended to other classes of (nonsoliton) solutions where typical catastrophe oscillations of the vortex string appear [7,8].

We will construct exact soliton solutions for (1.2) by following the procedure of the Inverse Scattering Transform (IST). The IST is a powerful method (see [9–13] for details) which allows one to solve the initial value problem for a class of nonlinear partial differential equations called *integrable equations*. The IST has already been applied to many significant nonlinear evolution equations such as the Korteweg–de Vries equation [14], the NLS equation [15], the cmKdV equation [16], and many other equations (see, for example, [17–19]), all of which can be derived from a suitable AKNS pair. Here by an integrable equation we mean an equation arising as a compatibility condition from an AKNS pair. An AKNS pair (see [20] for a detailed development of this subject) consists of two matrix functions X and T which depend on position, time, and the spectral variable λ not depending on x and t such that

$$\psi_x = X\psi \qquad \psi_t = T\psi. \tag{1.6}$$

The compatibility condition $\psi_{xt} = \psi_{tx}$ leads to the zero-curvature representation

$$X_t - T_x + XT - TX = 0$$

of a particular nonlinear partial differential equation.

We can look at (1.2) as a combination of the NLS equation and the cmKdV equation. Both of these equations are integrable and have, associated with them, the same operator (the so-called *Zakharov–Shabat* system (ZS) [15,21])

$$i\sigma_3 \frac{\partial \psi}{\partial x}(\lambda,x) - V(x)\psi(\lambda,x) = \lambda\psi(\lambda,x), \tag{1.7}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0 & iq(x) \\ ir(x) & 0 \end{pmatrix}, \tag{1.8}$$

λ is the spectral parameter and $q(x), r(x)$ are the potentials which, from now on, are supposed to belong to $L^1(\mathbb{R})$.

Also, the Hirota equation will have associated to it the Zakharov–Shabat system. Although the usual scattering theory of (1.7) has been developed for $q(x), r(x) \in L^1(\mathbb{R})$ (cf. [11,20]), the Hamiltonian formulation of the Hirota equation requires that $q(x), r(x)$ also belong to the first Sobolev space. By using this information, we are able to construct the scattering matrix and discover the form of the Marchenko equations. In this paper we consider only the so-called focusing case where $r(x) = q(x)^*$ (star denotes complex conjugate). In order to apply the IST to Eq. (1.2) we need to know how the scattering data thus obtained evolve in time. In Section 2 we will show how to discover which is the PDE satisfied by the Marchenko integral kernel of the corresponding Marchenko equations (which contains the scattering data). This is enough to solve the inverse scattering problem leading to the solution of Eq. (1.2).

To get the explicit (soliton) solution formula for (1.2) we apply the algebraic method recently developed and presented in [22–29]. The basic idea behind this method consists of representing the kernels of the Marchenko equations in a factorized form by using a triplet of matrices (A, B, C) and the matrix exponential in such a way that the Marchenko equations have separated variables. Then, these equations can be solved explicitly and their solutions are related to the solution of (1.2) (see formula (2.17)). Many authors studied the reflectionless solution of the Hirota equation. In the seminal paper [1] Hirota discovered the fact that the equation is bilinear and applied his method to find soliton solutions. In [30] the author studied the soliton solutions of the Hirota equations by means of the (iterative) Darboux transformation method and also rewrote the solution in the vortex filament context. Finally, in a recent paper [31], the authors extended the study to rogue wave solutions. However, to the best of our knowledge, there is no systematic analysis of these solutions. We provide, with this work, a complete solution formula using the IST method which is noniterative.

The second topic of this work is to find explicit time evolutions of a vortex filament associated with a specific soliton solution of (1.2). Using for the Hirota equation, the Sym–Pohlmeyer reconstruction formula [32–34] (see also [30,35] and [36] for closed filaments and [37] for the application to the full NLS hierarchy), we associate to a solution obtained via the IST procedure the explicit three dimensional motion of a vortex filament. The standard Hasimoto map connects these soliton solutions to the curvature and torsion of the curve but does not give the extrinsic motion of the curve as a map in $\mathbb{R}^3$. However, we are able to give the explicit expressions of curvature and torsion corresponding to a given soliton solution and we will see in Section 5 how to use this information to produce the graphics of the corresponding vortex filament.

We study explicitly some significant cases such as, for instance, the double soliton solution whose expression is obtained from the soliton solution formula. We also produce the plots of the corresponding vortex filament.

This paper is organized as follows: In Section 2 we briefly recall how the direct and inverse scattering problem for the Zakharov–Shabat system is usually studied in the literature. In Section 3 we derive a solution formula for Eq. (1.2) [in fact, formula (3.9)] when the reflection coefficient vanishes (soliton solution) by using the IST method. In Section 4, we discuss some (new) type of soliton solution obtained from formula (3.9) and plot their corresponding graphs. Finally, in Section 5 we get the explicit parametric equation of the surface of the vortex filament associated with a breather solution or a two soliton solution of (1.2). In Appendix A we give an independent proof of the validity of the formula found in Section 3.

2. Direct and inverse scattering theory for ZS system

In this section we recall the basic facts on the direct and inverse scattering theory of the ZS system and the IST method. The interested reader can find the proofs of the results presented here in [20] or, with slightly different notations, in [13,21].

Direct scattering theory of the ZS system. The direct scattering problem consists of constructing the scattering matrix $S(\lambda)$ which contains part of the scattering data. To this end, let us introduce the 2×1 columns known as *Jost functions from the right* $\overline{\psi}(\lambda, x)$ and $\psi(\lambda, x)$, the 2-component vectors known as *Jost functions from the left* $\phi(\lambda, x)$ and $\overline{\phi}(\lambda, x)$, and the 2×2 matrices called *Jost matrices* $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ *from the right and the left* as those solutions to the matrix the ZS system (1.7) satisfying the asymptotic conditions

$$\Psi(\lambda, x) = \begin{pmatrix}\overline{\psi}(\lambda, x) & \psi(\lambda, x)\end{pmatrix} = e^{-i\lambda\sigma_3 x}[I_2 + o(1)], \qquad x \to +\infty, \tag{2.1a}$$

$$\Phi(\lambda, x) = \begin{pmatrix}\phi(\lambda, x) & \overline{\phi}(\lambda, x)\end{pmatrix} = e^{-i\lambda\sigma_3 x}[I_2 + o(1)], \qquad x \to -\infty. \tag{2.1b}$$

where I_2 is the identity matrix of order 2. Using (2.1a) and (2.1b), we get the Volterra integral equations

$$\Psi(\lambda, x) = e^{-i\lambda\sigma_3 x} + i\sigma_3 \int_x^\infty dy\, e^{i\lambda\sigma_3(y-x)} V(y)\Psi(\lambda, y), \tag{2.2a}$$

$$\Phi(\lambda, x) = e^{-i\lambda\sigma_3 x} - i\sigma_3 \int_{-\infty}^x dy\, e^{-i\lambda\sigma_3(x-y)} V(y)\Phi(\lambda, y). \tag{2.2b}$$

Since the system of equations (1.7) is first order, we have

$$\Phi(\lambda, x) = \Psi(\lambda, x) a_r(\lambda), \qquad \Psi(\lambda, x) = \Phi(\lambda, x) a_l(\lambda). \tag{2.3}$$

We shall call $a_l(\lambda)$ and $a_r(\lambda)$ *transition matrices* from the left and the right, respectively; they are each other's inverses. From Eqs. (2.1) and (2.2), we get

$$\Psi(\lambda, x) = e^{-i\lambda\sigma_3 x}\left[a_l(\lambda) + o(1)\right], \quad x \to -\infty, \tag{2.4}$$

$$\Phi(\lambda, x) = e^{-i\lambda\sigma_3 x}\left[a_r(\lambda) + o(1)\right], \quad x \to +\infty. \tag{2.5}$$

It is more convenient to use the matrix representations

$$a_l(\lambda) = \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix}, \qquad a_r(\lambda) = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix},$$

where (cf. [11,13,20]) $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to 1 as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$. Here $\overline{\mathbb{C}^\pm}$ is the open upper/lower complex plane. In the same way we see that $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$, are analytic in $\lambda \in \mathbb{C}^-$, and tend to 1 as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^-}$. The remaining elements $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and vanish as $\lambda \to \pm\infty$.

The zeros $\lambda \in \mathbb{C}^+$ of $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$, are exactly the discrete eigenvalues of the system (1.7) in $\mathbb{C}^+$. On the other hand, the zeros $\lambda \in \mathbb{C}^-$ of $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are exactly the discrete eigenvalues of (1.7) in $\mathbb{C}^-$. We call $\lambda \in \mathbb{R}$ a *spectral singularity* if it is a zero of, at least, one of the diagonal elements $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$. In the sequel *we assume that there are no spectral singularities*. In that case, elementary complex analysis implies that the number of discrete eigenvalues of the system (1.7) is finite.

It is well-known [11,20] that for each $x \in \mathbb{R}$ the Jost functions $e^{-i\lambda x}\psi(\lambda, x)$ and $e^{i\lambda x}\phi(\lambda, x)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$, are analytic in $\lambda \in \mathbb{C}^+$, and behave as $\binom{0}{1}$ and as $\binom{1}{0}$ respectively, for $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$. Analogously, for each $x \in \mathbb{R}$ the Jost functions $e^{i\lambda x}\overline{\psi}(\lambda, x)$ and $e^{-i\lambda x}\overline{\phi}(\lambda, x)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$, are analytic in $\lambda \in \mathbb{C}^-$, and converge to $\binom{1}{0}$ and to $\binom{0}{1}$ respectively, as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^-}$. The above analyticity properties imply that for each $x \in \mathbb{R}$ the *modified Jost matrices* $F_\pm(\lambda, x)$ defined by

$$F_+(\lambda, x) = \begin{pmatrix}\phi(\lambda, x) & \psi(\lambda, x)\end{pmatrix} e^{i\lambda x\sigma_3}, \qquad F_-(\lambda, x) = \begin{pmatrix}\overline{\psi}(\lambda, x) & \overline{\phi}(\lambda, x)\end{pmatrix} e^{i\lambda x\sigma_3}, \tag{2.6}$$

are continuous in $\lambda \in \overline{\mathbb{C}^{\pm}}$, are analytic in $\mathbb{C}^{\pm}$, and converge to I_2 as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^{\pm}}$. The two modified Jost matrices are related as follows:

$$F_-(\lambda,x) = F_+(\lambda,x)\sigma_3 S(\lambda)\sigma_3, \qquad F_+(\lambda,x) = F_-(\lambda,x)\sigma_3 \breve{S}(\lambda)\sigma_3, \tag{2.7}$$

where the scattering matrices $S(\lambda)$ and $\breve{S}(\lambda)$ are each other's inverses. By writing them as

$$S(\lambda) = \begin{pmatrix} T(\lambda) & L(\lambda) \\ R(\lambda) & T(\lambda) \end{pmatrix}, \qquad \breve{S}(\lambda) = \begin{pmatrix} \breve{T}(\lambda) & \breve{R}(\lambda) \\ \breve{L}(\lambda) & \breve{T}(\lambda) \end{pmatrix},$$

we obtain the reflection coefficients $R(\lambda)$ and $\breve{R}(\lambda)$ from the right, the reflection coefficients $L(\lambda)$ and $\breve{L}(\lambda)$ from the left, the transmission coefficient $\breve{T}(\lambda)$ (which is meromorphic in $\lambda \in \mathbb{C}^-$), and the transmission coefficient $T(\lambda)$ (which is meromorphic in $\lambda \in \mathbb{C}^+$). Moreover, it is easily verified that

$$\breve{S}(\lambda) = \sigma_3 S(\lambda)^{\dagger}\sigma_3, \quad \text{for } \lambda \in \mathbb{R}.$$

where the $\dagger$ denotes matrix transpose conjugation. Under the assumption that there are no spectral singularities, we also have

$$R(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\rho(y), \qquad L(\lambda) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y}\ell(y), \tag{2.8a}$$

where ρ, ℓ belong to $L^1(\mathbb{R})$. Furthermore, $\breve{R}(\lambda)$ and $\breve{L}(\lambda)$ have analogous representations where $\breve{\rho} = -\rho(y)^*$, $\breve{\ell} = -\ell(y)^*$ replace ρ, ℓ. The scattering data associated with (1.7) consist of one reflection coefficient, the discrete eigenvalues of (1.7) and a suitable set of positive constants associated to them (the so-called norming constants). The construction of the norming constants can be found in [11] (where the case when all the eigenvalues have algebraic multiplicity one is considered) or in [24,38–40] (where the more general case is treated).

Inverse scattering theory of the ZS system. The inverse scattering problem consists of the (re)-construction of the (unique) potential $q(x)$ if the scattering data are given. Following [13,24,25], we formulate and solve this problem by using the Marchenko method (see also [15,41]). Writing the Fourier representations

$$\Psi(\lambda,x) = \begin{pmatrix} \overline{\psi}(\lambda,x) & \psi(\lambda,x) \end{pmatrix} = e^{-i\lambda\sigma_3 x} + \int_x^{\infty} dy\, \alpha_l(x,y)e^{-i\lambda\sigma_3 y}, \tag{2.9a}$$

$$\Phi(\lambda,x) = \begin{pmatrix} \overline{\phi}(\lambda,x) & \phi(\lambda,x) \end{pmatrix} = e^{-i\lambda\sigma_3 x} + \int_{-\infty}^{x} dy\, \alpha_r(x,y)e^{-i\lambda\sigma_3 y}, \tag{2.9b}$$

we obtain, in a well-known way [11,20], the Marchenko integral equations

$$\alpha_l(x,y) + \omega_l(x+y) + \int_x^{\infty} dz\, \alpha_l(x,z)\omega_l(z+y) = 0_{2\times 2}, \tag{2.10a}$$

$$\alpha_r(x,y) + \omega_r(x+y) + \int_{-\infty}^{x} dz\, \alpha_r(x,z)\omega_r(z+y) = 0_{2\times 2}, \tag{2.10b}$$

where, for later use, we introduced the notations

$$\alpha_l(x,y) = \begin{pmatrix} \overline{K}(x,y) & K(x,y) \end{pmatrix}, \qquad \alpha_r(x,y) = \begin{pmatrix} M(x,y) & \overline{M}(x,y) \end{pmatrix} \tag{2.11}$$

and $\overline{K}(x,y)$, $K(x,y)$, $M(x,y)$, $\overline{M}(x,y)$ are column vectors of length two (up and down will denote the first and second components of such column vectors). Furthermore, $\omega_l(x+y)$, $\omega_r(x+y)$ are called the left and right *Marchenko kernels*, respectively. These kernels anticommute with σ_3 in the sense that

$$\sigma_3\omega_l(y+z) = -\omega_l(y+z)\sigma_3, \qquad \sigma_3\omega_r(y+z) = -\omega_r(y+z)\sigma_3,$$

and satisfy $\omega_{l/r}(y+z)^{\dagger} = \sigma_3\omega_{l/r}(y+z)\sigma_3$. It is well known that these kernels are given by

$$\omega_l(x) = \begin{pmatrix} 0 & -\rho(x)^* - \sum_{j=1}^{\breve{N}}\sum_{s=0}^{j} \frac{x^s}{s!} e^{-x\lambda_j^*}[C_l]_{js}^* \\ \rho(x) + \sum_{j=1}^{N}\sum_{s=0}^{j} \frac{x^s}{s!} e^{-x\lambda_j}[C_l]_{js} & 0 \end{pmatrix}, \tag{2.12}$$

$$\omega_r(x) = \begin{pmatrix} 0 & \ell(x) + \sum_{j=1}^{N}\sum_{s=0}^{j} \frac{x^s}{s!} e^{x\lambda_j}[C_r]_{js} \\ -\ell(x)^* - \sum_{j=1}^{\breve{N}}\sum_{s=0}^{j} \frac{x^s}{s!} e^{x\lambda_j^*}[C_r]_{js}^* & 0 \end{pmatrix}, \tag{2.13}$$

where λ_j are the discrete eigenvalues belonging to the upper (lower) half plane, and $[C_{lr}]_{js}$ are the norming constants associated with the discrete eigenvalues.

In general, for $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the potential $q(x)$ is related to the Marchenko solutions $\alpha_l(x, y)$ and $\alpha_r(x, y)$ as follows (cf. (A.2) and (A.4) in [29]):

$$\alpha_l(x,x) = -\frac{1}{2}\begin{pmatrix} \int_x^\infty dz\, |q(z)|^2 & q(x) \\ -q(x)^* & \int_x^\infty dz\, |q(z)|^2 \end{pmatrix}, \tag{2.14a}$$

$$\alpha_r(x,x) = -\frac{1}{2}\begin{pmatrix} \int_{-\infty}^x dz\, |q(z)|^2 & -q(x) \\ q(x)^* & \int_{-\infty}^x dz\, |q(z)|^2 \end{pmatrix}. \tag{2.14b}$$

As a result, we can recover the potential $q(x)$ following the three steps indicated below:

a. Suppose that the reflection coefficient $R(\lambda)$, the discrete eigenvalues $\{\lambda_j\}_{j=1}^N$ and the norming constants $\left\{\{C_{js}\}_{s=0}^{n_j-1}\right\}_{j=1}^N$ are given, where N denotes the number of discrete eigenvalues in $\mathbb{C}^+$, while n_j is the multiplicity of λ_j. By using the scattering data we introduce the function

$$\Omega_l(y) \overset{\text{def}}{=} -\rho^\dagger(y) + \sum_{j=1}^N \sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y}, \tag{2.15}$$

where $\rho(y) = \frac{1}{2\pi}\int_{-\infty}^\infty R(\lambda) e^{i\lambda y} d\lambda$ is the inverse Fourier transform of $R(\lambda)$.

b. Solve the following integral equation Marchenko:

$$K^{up}(x,y) - \Omega_l(x+y)^\dagger + \int_x^\infty dz \int_x^\infty ds\, K^{up}(x,z)\, \Omega_l(z+s)\, \Omega_l(s+y)^\dagger = 0, \tag{2.16}$$

where $y > x$.

c. Finally, the potential $q(x)$ is obtained by using the following formula:

$$q(x) = -2K^{up}(x,x). \tag{2.17}$$

An analogous procedure can be followed by using the right Marchenko kernel.

Time evolution of the scattering data. If one knows the operators X and T in the compatibility problem (1.6) related to the Hirota equation, it is easy to find the scattering data and their time evolution (see [9]). As underlined in the introduction, the Hirota equation can be considered as the sum of two flows belonging to the same hierarchy. The matrix X generally depends only on the hierarchy but not on the particular equation. Therefore, for the Hirota equation, the matrix X is the same one which appears in the NLS compatibility problem. We determine the matrix T as follows. Let us denote with $T^{(1)}$ ($T^{(2)}$) the matrix related to the time evolution operators of the focusing NLS (cmKdV) equation given, respectively, by

$$X = -i\lambda\sigma_3 + \tilde{V}, \tag{2.18}$$

$$T^{(1)} = -2i\lambda^2\sigma_3 + 2\lambda\tilde{V} + i\sigma_3(\tilde{V}_x - \tilde{V}^2), \tag{2.19}$$

$$T^{(2)} = -4i\lambda^3\sigma_3 + 4\lambda^2\tilde{V} + 2i\lambda\sigma_3(\tilde{V}_x - \tilde{V}^2) + (-\tilde{V}_{xx} + 2\tilde{V}^3 + [\tilde{V}_x, \tilde{V}]). \tag{2.20}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{V} = -i\sigma_3 V = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{2.21}$$

We have

$$\begin{cases} \psi_x^{(s)} = X\psi^{(s)} \\ \psi_t^{(s)} = T^{(s)}\psi^{(s)} \end{cases} \tag{2.22}$$

for $s = 1$ or $s = 2$ and the case $s = 1$ ($s = 2$) refer to the AKNS pair of the NLS (cmKdV) equation. Furthermore, we get the following zero-curvature conditions:

$$X_t - T_x^{(1)} = XT^{(1)} - T^{(1)}X, \tag{2.23}$$

$$X_t - T_x^{(2)} = XT^{(2)} - T^{(2)}X, \tag{2.24}$$

where (2.23) is the zero-curvature condition for the NLS equation, while (2.24) is the zero-curvature condition for the cmKdV equation. Defining T as

$$T = -\alpha_2 T^{(1)} + \alpha_3 T^{(2)}, \tag{2.25}$$

where α_2 and α_3 are real constants, and rescaling the time variable as $\tau = \frac{t}{-\alpha_2+\alpha_3}$ we arrive at the following zero-curvature condition:

$$X_\tau - T_x = XT - TX, \tag{2.26}$$

where α_2, α_3 appear in (1.2). It is worth noting that the two main special cases (focusing NLS and cmKdV) satisfy $\alpha_3 - \alpha_2 = 1$ where the time rescaling is not necessary.

We are interested in constructing soliton solutions. This type of solution is characterized by the condition $R(\lambda) = 0$. So, the kernel which appears in the Marchenko equation (2.16) when we take into account the evolution of the scattering data, is given by

$$\Omega_l(y;t) = \sum_{j=1}^{N}\sum_{s=0}^{n_j-1} c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y}. \tag{2.27}$$

In the literature the time evolution of the kernel $\Omega_l(y,t)$ for both NLS and cmKdV is well-known. By using the ideas present in e.g. [20] it is easy to see that the construction of the kernel is linear in the transmission and reflection coefficients. As a result, taking into account (2.25), we obtain

$$\Omega_{lt} - 4i\alpha_2\Omega_{lyy} + 8\alpha_3\Omega_{lyyy} = 0. \tag{2.28}$$

Inverse Scattering Transform. Having presented the direct and inverse scattering problems corresponding to the ZS system and the time evolution of the scattering data, we can discuss how the IST allows us to obtain the solution to the initial value problem for (1.2).

Using the initial condition $q(x,0)$ as a potential in the system (1.7), we develop the direct scattering theory as shown above and build the scattering data. Successively, let the initial scattering data evolve in time in agreement with Eq. (2.28). The solution of the Hirota equation is then obtained by solving the Marchenko equation (2.16) where the kernel $\Omega_l(y)$ is replaced by $\Omega_l(y;t)$, and then using relation (2.17).

3. Soliton solutions of the Hirota equation

In this section we construct an explicit soliton solution formula for Eq. (1.2). We apply the same technique successfully used in [22,24,25] to solve, respectively, the NLS, the sine–Gordon, and the cmKdV equation. The basic idea behind this method is to represent the kernel appearing in the Marchenko equation in a separated form. This leads to explicitly solvable Marchenko equations and then, by using Eq. (2.17), we can derive an explicit solution formula for Eq. (1.2).

We recall that we want to investigate the case $R(\lambda) = 0$. Then the general expression for $\Omega_l(y;0)$ is given by (2.27). The discrete eigenvalues terms can be written in the form (see [25] for more details on this representation of the scattering data)

$$\Omega_l(y;t) = \sum_{j=1}^{N}\sum_{s=0}^{n_j-1} c_{js}(t)\frac{y^s}{s!}e^{i\lambda_j y} = C(t)e^{-yA}B, \tag{3.1}$$

where $\lambda_1,\ldots,\lambda_N$ are the discrete eigenvalues, n_j are the orders of the poles of the transmission coefficient at the discrete eigenvalues $i\lambda_j$, and c_{js} are the so-called norming constants. Here (A,B,C) is a triplet of matrices of order $p\times p$, $p\times 1$, $p\times 1$, respectively, where p is a positive integer number and C depends on t. Moreover, for reasons which will be clear later, we need to put some suitable properties on this class of triplets. In fact, we require that

1. The eigenvalues of the matrix A have positive real parts;
2. The triplet (A,B,C) provides a minimal representation for the kernel $\Omega_l(y;t)$, which means that

$$\bigcap_{r=1}^{+\infty}\ker CA^{r-1} = \bigcap_{r=1}^{+\infty}\ker B^\dagger(A^\dagger)^{r-1} = \{0\},$$

(we refer the reader to [42,43] for more details on minimal representations).

On the other hand, the evolved kernel $\Omega_l(y;t)$ has to satisfy Eq. (2.28). It is easy to verify that Eq. (2.28) is satisfied if we take $\Omega_l(y;t)$ as

$$\Omega_l(y;t) = Ce^{-i\phi(iA)t}e^{-yA}B, \tag{3.2}$$

where

$$\phi(z) = 4\alpha_2 z^2 - 8\alpha_3 z^3. \tag{3.3}$$

Then Eqs. (3.2)–(3.3) give us the time evolution of the kernel $\Omega_l(y;t)$.

In order to derive our soliton solution formula, we have to solve the Marchenko equation (2.16) where $\Omega_l(y)$ is replaced by $\Omega_l(y;t)$. Substituting the expression (3.2) in Eq. (2.16) and looking for a solution in the form

$$K^{up}(x,y;t) = H(x,t)e^{-A^\dagger y + i\phi(-iA^\dagger)t}C^\dagger, \tag{3.4}$$

we arrive at the equation

$$H(x;t) + H(x;t)\int_x^\infty dz\int_x^\infty ds\, e^{-A^\dagger z + i\phi(-iA^\dagger)t}C^\dagger Ce^{-Az - i\phi(iA)t}e^{-As}BB^\dagger e^{-A^\dagger s} = B^\dagger e^{-A^\dagger x}. \tag{3.5}$$

Introducing the $p\times p$ matrices Q and N as

$$Q = \int_0^\infty ds\, e^{-A^\dagger s}C^\dagger Ce^{-As}, \qquad N = \int_0^\infty dr\, e^{-Ar}BB^\dagger e^{-A^\dagger r}, \tag{3.6}$$

after some easy calculations we obtain

$$H(x,t)\Gamma(x,t) = B^\dagger e^{-A^\dagger x}, \tag{3.7}$$

where

$$\Gamma(x,t) = I_p + e^{-A^\dagger x + i\phi(-iA^\dagger)t} Q e^{-2Ax - i\phi(iA)t} N e^{-A^\dagger x}, \tag{3.8}$$

and I_p the identity matrix of order p. Finally, by using Eq. (3.4) and relation (2.17) we get the following soliton solution formula for Eq. (1.2):

$$q(x,t) = -2B^\dagger e^{-A^\dagger x} \Gamma^{-1}(x,t) e^{-A^\dagger x + i\phi(-iA^\dagger)t} C^\dagger. \tag{3.9}$$

In Appendix A we give another independent direct proof that the function $q(x,t)$ given by Eq. (3.9) satisfies the Hirota equation (1.2). We observe that our solution formula depends only on the matrix triplet used as input. In fact, given the triplet of matrices, we can build the matrices Q, N and $\Gamma(x,t)$ by using formulas (3.6) and (3.8), respectively, and then we can easily write the solution formula (3.9). However, we observe that the solution (3.9) exists only if for all $(x,t) \in \mathbb{R}^2$ the integrals (3.6) converge and the matrix $\Gamma(x,t)$ is invertible. So we have to establish when the integrals (3.6) converge and the matrix $\Gamma(x,t)$ is invertible. To do so, let us introduce the following notations:

$$\overline{P}(x,t) = \int_x^\infty ds\, e^{-A^\dagger s + i\phi(-iA^\dagger)t} C^\dagger C e^{-As - i\phi(iA)t}, \tag{3.10}$$

$$P(x) = \int_x^\infty dr\, e^{-Ar} B B^\dagger e^{-A^\dagger r}. \tag{3.11}$$

Then

$$\Gamma(x,t) = I_p + \overline{P}(x,t) P(x). \tag{3.12}$$

The following proposition justifies the requirement that the eigenvalues of matrix A have positive real parts:

Proposition 3.1. *The matrices $\overline{P}(x;t)$ and $P(x)$ defined in (3.10) and (3.11), respectively, satisfy*

$$\overline{P}(x;t) = e^{-A^\dagger x + i\phi(-iA^\dagger)t} Q e^{-Ax - i\phi(iA)t}, \qquad P(x) = e^{-Ax} N e^{-A^\dagger x}$$

and the integrals in (3.10) and (3.11) converge for all $(x,t) \in \mathbb{R}$, provided the eigenvalues of A have positive real parts.

Proof. Replacing the time factor $e^{-4iA^2 t}$ with $e^{i\phi(-iA)t}$ where $\phi(-iA)$ is given by (3.3), the proof of this proposition can be obtained by repeating the proof of Proposition 4.1 in [24] verbatim. ■

Moreover, we also have

Proposition 3.2. *Suppose that the eigenvalues of the matrix A have positive real parts. Then, for every $(x,t) \in \mathbb{R}$ the matrices $\overline{P}(x,t)$, $P(x)$ and $\Gamma(x,t)$ satisfy the following properties:*

(a) *The matrices $\overline{P}(x,t)$, $P(x)$ are selfadjoint.*
(b) *The matrix $\Gamma(x,t)$ is invertible.*
(c) *The matrices $\overline{P}(x,t)$, $P(x)$ are the unique solutions of the Lyapunov equations*

$$A^\dagger \overline{P}(x,t) + \overline{P}(x,t) A = e^{i\phi(-iA^\dagger)t} C^\dagger C e^{-i\phi(iA)t}, \tag{3.13}$$

$$AP + PA^\dagger = BB^\dagger. \tag{3.14}$$

Proof. The proof of the points (a) and (b) of this proposition is identical to the proof of Theorem 4.2 in [24]. A proof of (c) can be found in [20,43]. ■

The following proposition shows why it is important to make the hypothesis of minimality on the triplet (A,B,C).

Proposition 3.3. *Suppose that the eigenvalues of the matrix A have positive real parts and that the triplet (A,B,C) is minimal. Then, for each fixed t, $\Gamma(x;t)^{-1} \to I_p$ as $x \to +\infty$ and $\Gamma(x;t)^{-1} \to 0$ as $x \to -\infty$.*

Proof. The proof of this proposition is identical to the proof of Proposition 4.6 in [24]. ■

We remark that the proof of the statement $\Gamma(x;t)^{-1} \to 0$ as $x \to -\infty$ requires the hypothesis of minimality of the triplet. Proposition 3.3 is important because from it we immediately get the following.

Proposition 3.4. *Suppose that the eigenvalues of the matrix A have positive real parts and that the triplet (A,B,C) is minimal. Then the scalar function $q(x,t)$ decays exponentially for each fixed t as $x \to \pm\infty$.*

It is natural to look for a larger class of triplets of matrices in such a way that the formula (3.9) holds, which means that the integrals in (3.6) converge and the inverse of the matrix $\Gamma(x,t)$ exists for all $(x,t) \in \mathbb{R}^2$. This was accomplished in [44] where the so-called *admissible class* of matrix triplets has been introduced. Without giving the details, the main result is synthesized by the next proposition which allows us to understand the "canonical way" to choose the triplet (A,B,C) in (3.9).

Two triplets $(\tilde{A}, \tilde{B}, \tilde{C})$ and (A,B,C) in the admissible class are called *equivalent* if they lead to the same potential $q(x,t)$ (given by formula (3.9)).

Proposition 3.5. *Starting from $(\tilde{A}, \tilde{B}, \tilde{C})$ in the admissible class, it is possible to associate to this triplet an equivalent triplet (A, B, C) where A has the Jordan canonical form with each Jordan block containing a distinct eigenvalue having a positive real part, the column B consists of zeros and ones, and C has real entries. More specifically, for some appropriate positive integer m, we have*

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_2 & \cdots & C_m \end{pmatrix}, \tag{3.15}$$

where in the case of a real (positive) eigenvalue ω_j of A_j the corresponding blocks are given by

$$A_j := \begin{pmatrix} \omega_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_j & -1 & \cdots & 0 & 0 \\ 0 & 0 & \omega_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{3.16}$$
$$C_j := \begin{pmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{pmatrix},$$

A_j having size $n_j \times n_j$, B_j size $n_j \times 1$, C_j size $1 \times n_j$, and the constant c_{jn_j} is nonzero. In the case of complex eigenvalues, which must appear in pairs as $\alpha_j \pm i\beta_j$ with $\alpha_j > 0$, the corresponding blocks are given by

$$A_j := \begin{pmatrix} \Lambda_j & -I_2 & 0 & \dots & 0 & 0 \\ 0 & \Lambda_j & -I_2 & \dots & 0 & 0 \\ 0 & 0 & \Lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Lambda_j & -I_2 \\ 0 & 0 & 0 & \dots & 0 & \Lambda_j \end{pmatrix}, \qquad B_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{3.17}$$
$$C_j := \begin{pmatrix} \gamma_{jn_j} & \epsilon_{jn_j} & \dots & \gamma_{j1} & \epsilon_{j1} \end{pmatrix},$$

where γ_{js} and ϵ_{js} for $s = 1, \dots, n_j$ are real constants with $(\gamma_{jn_j}^2 + \epsilon_{jn_j}^2) > 0$, each column vector B_j has $2n_j$ components, each A_j has size $2n_j \times 2n_j$, and the 2×2 matrix Λ_j is defined as

$$\Lambda_j := \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}. \tag{3.18}$$

Proof. The triplet (A, B, C) can be chosen as in Section 3 of [24]. ■

The complete analysis of the possible choices of the matrices is also useful for the study of the α_3-effects on the reflectionless solutions. This coefficient appears in the real exponent of the Γ matrix in the ϕ function evaluated at the constant matrix A. The diagonal part of A gives a real exponential contribution to the final solution (3.9). This new exponent contribution is asymptotically responsible for the possibility to choose arbitrarily the center of mass[1] of the solution. If A is non-diagonalizable the off-diagonal terms of the Jordan block representation yield a polynomial (in t) contribution to the solution. As we will see more explicitly in the two-poles example, this affects the separation velocity of the peaks. In the last section we will present the filament counterpart of these properties which can be easily obtained from (3.9).

4. Examples

In Proposition 3.5 we have classified the possible inequivalent classes of triplets (A, B, C) used for the construction of the soliton solutions in (3.9). Such classes are expressed in terms of the component blocks A_j appearing in (3.16). To any block corresponds a qualitatively different behavior of the solution as shown in Table 1.

In this section we concentrate our attention on a comparison between the well known class of soliton solutions and the class of multipole solutions found in the previous section. To show the differences we use the simple example of the two-poles solution of the Hirota equation. We reproduce also the standard two soliton solutions in order to stress the qualitative differences with the two-pole soliton solutions which can be regarded as a limiting case of the soliton behavior.

In all these examples, to get the explicit expressions of the solutions we write (3.9) in the following way:

$$q(x, t) = \frac{-2B^\dagger \left[\text{cofac}\Delta(x; t)\right] \left[Ce^{-2xA}e^{t\mathbb{A}}\right]^\dagger}{\det \Delta(x; t)}, \tag{4.1}$$

[1] As usual the center of mass X_f of a function f is defined as

$$X_f = \frac{\int_{\mathbb{R}} xf(x)\mathrm{d}x}{\int_{\mathbb{R}} f(x)\mathrm{d}x}.$$

Table 1
Examples of soliton behavior as a function of the matrix A in the triplet (A, B, C).

Block A_j in (3.15)	Solution behavior
1×1 real matrix	1-soliton solution with $\|q(x,t)\| = f(x)$
1×1 complex matrix	1-soliton solution with $\|q(x,t)\| = f(x - vt)$
2×2 matrix with complex conjugate eigenvalues	1-breather solution ("2-particle bound state")
Jordan block (3.16) of order s	Multipole solution

where

$$\Delta(x;t) \stackrel{\text{def}}{=} e^{-xA^\dagger}\Gamma(x;t)e^{xA^\dagger} = I_2 + e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger}Qe^{-2xA}e^{t\mathbb{A}}N, \quad \mathbb{A} = 4i\alpha_2A^2 + 8\alpha_3A^3,$$

while Q and N are the solutions of the Lyapunov equations (3.14). In other words, we need to calculate $\det\Delta(x,t)$ and the inverse of $\Delta(x,t)$.

2-soliton solution. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & -2 \end{pmatrix}.$$

Then it is easily verified that

$$Q = \begin{pmatrix} \frac{9}{4} & -2 \\ -2 & 2 \end{pmatrix}, \qquad N = \begin{pmatrix} \frac{9}{4} & 2 \\ 2 & 2 \end{pmatrix},$$

are the unique solutions to the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \qquad AN + NA^\dagger = BB^\dagger.$$

Let us compute the time factor $e^{t\mathbb{A}}$ where

$$\mathbb{A} = 4i\alpha_2A^2 + 8\alpha_3A^3 = \begin{pmatrix} 16i\alpha_2 + 64\alpha_3 & 0 \\ 0 & 4i\alpha_2 + 8\alpha_3 \end{pmatrix}.$$

Then

$$e^{-2xA}e^{t\mathbb{A}} = \begin{pmatrix} e^{-4(x-dt)} & 0 \\ 0 & e^{-2(x-et)} \end{pmatrix}, \qquad e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger} = \begin{pmatrix} e^{-4(x-d^*t)} & 0 \\ 0 & e^{-2(x-e^*t)} \end{pmatrix},$$

where $d = 4i\alpha_2 + 16\alpha_3$ and $e = 2i\alpha_2 + 4\alpha_3$. Therefore,

$$e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger}Q = \begin{pmatrix} \frac{9}{4}e^{-4(x-d^*t)} & -2e^{-4(x-d^*t)} \\ -2e^{-2(x-e^*t)} & 2e^{-2(x-e^*t)} \end{pmatrix}, \qquad e^{-2xA}e^{t\mathbb{A}}N = \begin{pmatrix} \frac{9}{4}e^{-4(x-dt)} & 2e^{-4(x-dt)} \\ 2e^{-2(x-et)} & 2e^{-2(x-et)} \end{pmatrix}.$$

As a result,

$$\begin{aligned}\Delta(x;t) &= I_2 + e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger}Qe^{-2xA}e^{t\mathbb{A}}N \\ &= \begin{pmatrix} 1 + \frac{81}{16}e^{-8(x-16\alpha_3t)} - 4e^{-6(x-f^*t)} & \frac{9}{2}e^{-8(x-16\alpha_3t)} - 4e^{-6(x-f^*t)} \\ -\frac{9}{2}e^{-6(x-ft)} + 4e^{-4(x-4\alpha_3)t} & 1 - 4e^{-6(x-ft)} + 4e^{-4(x-4\alpha_3t)} \end{pmatrix},\end{aligned}$$

where $f = \frac{1}{6}(4d + 2e^*) = 2i\alpha_2 + 12\alpha_3$. Consequently,

$$\det\Gamma(x;t) = 1 + \tfrac{81}{16}e^{-8(x-16\alpha_3t)} + 4e^{-4(x-4\alpha_3t)} - 8e^{-6(x-12\alpha_3t)}\cos(12\alpha_2t) + \frac{1}{4}e^{-12(x-12\alpha_3t)},$$

which obviously exceeds 1.

Next,

$$\begin{aligned}[\det\Gamma(x;t)]q(x,t) &= -2B^\dagger[\operatorname{cofac}\Delta(x;t)]\left[Ce^{-2xA}e^{t\mathbb{A}}\right]^\dagger \\ &= -18e^{-4(x-d^*t)}\left[1 - 4e^{-6(x-ft)} + 4e^{-4(x-4\alpha_3t)}\right] + 12e^{-4(x-d^*t)}\left[\tfrac{-9}{2}e^{-6(x-ft)} + 4e^{-4(x-4\alpha_3t)}\right] \\ &\quad + 8e^{-2(x-e^*t)}\left[1 + \tfrac{81}{16}e^{-8(x-16\alpha_3t)} + 8e^{-6(x-f^*t)}\right] - 12e^{-2(x-e^*t)}\left[\tfrac{9}{2}e^{-8(x-16\alpha_3t)} - 4e^{-6(x-f^*t)}\right].\end{aligned} \tag{4.2}$$

We remark that choosing $\alpha_2 = -1$ and $\alpha_3 = 0$, i.e., when the Hirota equation reduces at the focusing NLS equation, we get the solution

$$q(x,t) = \frac{8e^{4it}\left(9e^{-4x} + 16e^{4x}\right) - 32e^{16it}\left(4e^{-2x} + 9e^{2x}\right)}{-128\cos(12t) + 4e^{-6x} + 16e^{6x} + 81e^{-2x} + 64e^{2x}}.$$

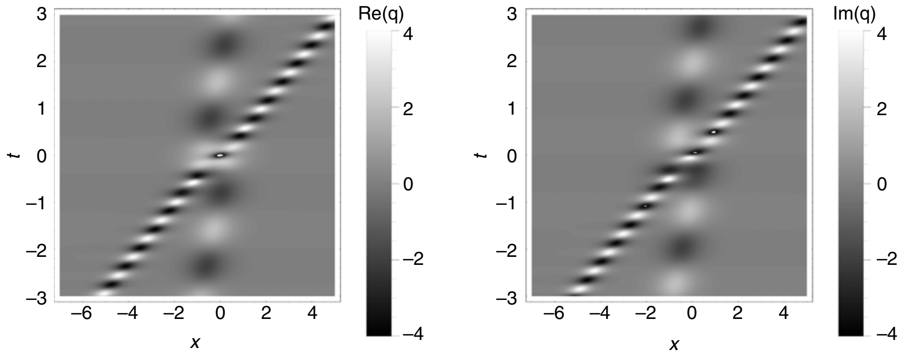

Fig. 4.1. An example of a two-soliton solution in the case $\alpha_2 = -1$, $\alpha_3 = 0.1$. The matrix triplet is given by (4.3).

This solution coincides exactly with the 2-soliton solution obtained in [44] for the NLS equation by using the same triplet of matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & -2 \end{pmatrix}. \tag{4.3}$$

A plot of the solution can be found in Fig. 4.1.

Double pole solution.

Let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{4.4}$$

Then it is easily verified that

$$Q = \tfrac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad N = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

are the unique solutions to the Lyapunov equations. Clearly,

$$e^{-2xA} = e^{-2x}\begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}, \qquad e^{-2xA^\dagger} = e^{-2x}\begin{pmatrix} 1 & 0 \\ 2x & 1 \end{pmatrix}.$$

Let us calculate the time factor $e^{t\mathbb{A}}$, where

$$\mathbb{A} = 4i\alpha_2 A^2 + 8\alpha_3 A^3 = (4i\alpha_2 + 8\alpha_3)I_2 - (8i\alpha_2 + 24\alpha_3)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$e^{t\mathbb{A}} = e^{(4i\alpha_2+8\alpha_3)t}\begin{pmatrix} 1 & -(8i\alpha_2 + 24\alpha_3)t \\ 0 & 1 \end{pmatrix},$$
$$e^{t\mathbb{A}^\dagger} = e^{(-4i\alpha_2+8\alpha_3)t}\begin{pmatrix} 1 & 0 \\ -(-8i\alpha_2 + 24\alpha_3)t & 1 \end{pmatrix}.$$

Putting $d = 2i\alpha_2 + 4\alpha_3$ and $e = 4i\alpha_2 + 12\alpha_3$ (so that $\frac{d+d^*}{2} = 4\alpha_3$ and $\frac{e+e^*}{2} = 12\alpha_3$), we get

$$e^{-2xA}e^{t\mathbb{A}} = e^{-2(x-dt)}\begin{pmatrix} 1 & 2(x-et) \\ 0 & 1 \end{pmatrix},$$
$$e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger} = e^{-2(x-d^*t)}\begin{pmatrix} 1 & 0 \\ 2(x-e^*t) & 1 \end{pmatrix}.$$

We obtain

$$e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger}Q = \frac{e^{-2(x-d^*t)}}{4}\begin{pmatrix} 2 & 1 \\ 4(x-e^*t)+1 & 2(x-e^*t)+1 \end{pmatrix},$$
$$e^{-2xA}e^{t\mathbb{A}}N = \frac{e^{-2(x-dt)}}{4}\begin{pmatrix} 2(x-et)+1 & 4(x-et)+1 \\ 1 & 2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\Delta(x;t) &= I_2 + e^{-2xA^\dagger}e^{t\mathbb{A}^\dagger}Qe^{-2xA}e^{t\mathbb{A}}N \\ &= I_2 + \frac{e^{-4(x-4\alpha_3 t)}}{16}\begin{pmatrix} 4(x-et)+3 & 8(x-et)+4 \\ 8|x-et|^2 + 2(x-et) + 6(x-e^*t) + 2 & 16|x-et|^2 + 4(x-et) + 8(x-e^*t) + 3 \end{pmatrix}.\end{aligned}$$

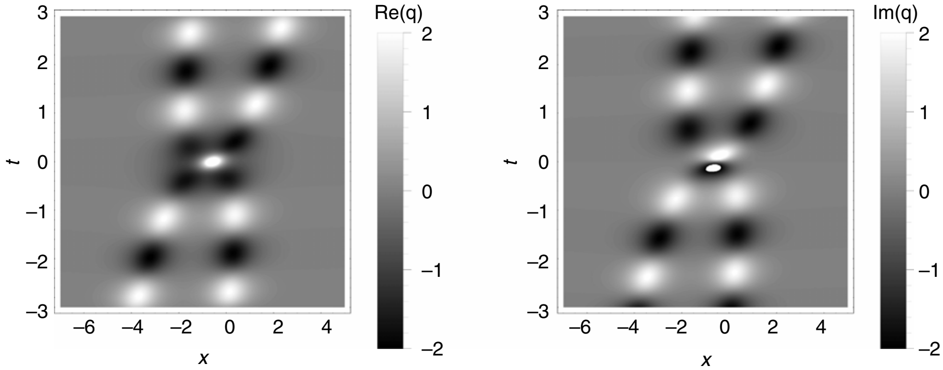

Fig. 4.2. An example of a double pole solution in the case $\alpha_2 = -1, \alpha_3 = 0.1$. The matrix triplet given by (4.4).

Consequently,

$$\det \Gamma(x;t) = 1 + \frac{e^{-4(x-4\alpha_3 t)}}{16}\left\{16|x-et|^2 + 8(x-et) + 8(x-e^*t) + 6\right\} + \frac{e^{-8(x-4\alpha_3 t)}}{256}$$
$$= 1 + e^{-4(x-4\alpha_3 t)}\left\{(x - 12\alpha_3 t + \tfrac{1}{2})^2 + 16\alpha_2^2 t^2 + \tfrac{1}{8}\right\} + \frac{e^{-8(x-4\alpha_3 t)}}{256},$$

which is larger than 1.

Next,

$$[\det \Gamma(x;t)]q(x,t) = -2B^\dagger[\mathrm{cofac}\Delta(x;t)]\left[Ce^{-2xA}e^{t\mathbb{A}}\right]^\dagger$$
$$= -4e^{-2(x-d^*t)}\left\{x - e^*t - \tfrac{1}{16}(x-et+1)e^{-4(x-4\alpha_3 t)}\right\}.$$

Consequently,

$$q(x,t) = -4e^{-2(x-d^*t)}\frac{x - e^*t - \frac{1}{16}(x-et+1)e^{-4(x-4\alpha_3 t)}}{1 + e^{-4(x-4\alpha_3 t)}\left\{(x-12\alpha_3 t + \frac{1}{2})^2 + 16\alpha_2^2 t^2 + \frac{1}{8}\right\} + \frac{e^{-8(x-4\alpha_3 t)}}{256}}, \tag{4.5}$$

where $d^* = -2i\alpha_2 + 4\alpha_3$, $e = 4i\alpha_2 + 12\alpha_3$, and $e^* = -4i\alpha_2 + 12\alpha_3$. The solution is plotted in Fig. 4.2. The main difference between two-soliton and two-pole solutions relies on the asymptotic behavior of the two bumps present. As it is well known the two-soliton solution Fig. 4.1 of the Hirota equation is asymptotically given by two separate bumps of different height and velocity. In the two-pole case Fig. 4.2 the bumps are asymptotically of the same height and the separation speed is much slower. In the simple NLS case ($\alpha_2 = -1, \alpha_3 = 0$) this result is known from [15] where the authors remark that the separation of the two peaks is logarithmic in time. We can find the analogue of these remarks in the full Hirota case. Let us rewrite the double pole solution (4.5) as

$$q(x,t) = -\frac{4e^{-4i\alpha_2 t + 8\alpha_3 t - 2x}\left(4it(\alpha_2 + 3i\alpha_3) - \frac{1}{16}e^{16\alpha_3 t - 4x}(-4i\alpha_2 t - 12\alpha_3 t + x + 1) + x\right)}{e^{16\alpha_3 t - 4x}\left(16{\alpha_2}^2 t^2 + \left(-12\alpha_3 t + x + \frac{1}{2}\right)^2 + \frac{1}{8}\right) + \frac{1}{256}e^{32\alpha_3 t - 8x} + 1}. \tag{4.6}$$

As one can see directly from the explicit form of the solution, there are two contributions of α_3. Indeed, actually this constant is present both in the real exponent and in some mononomial terms in t. In the exponent (the dominant effect) x appears always in the combination $4\alpha_3 t - x$. Therefore the center of mass of the (real and imaginary part) of the solution moves asymptotically for large t at the speed $4\alpha_3$. The monomial terms containing α_3 in the coefficients simply give a quantitative contribution to the logarithmic relative velocity of the two peaks. We remark that this solution is valid for every value of α_3 even if it is larger than α_2, and also in the limiting cmKdV-case $\alpha_2 = 0$.

5. Vortex filaments

In this section we apply the results obtained so far to write down explicitly the vortex filament evolution satisfying (1.4) associated with a (specific) soliton solution of the Hirota equation. Moreover, we use the explicit solution formula (3.9) to represent such vortex filament in terms of its curvature and torsion.

We recall that [32–34] the cartesian components $x_i(x,t)$, (for $i = 1, 2, 3$) of the curve (for a fixed t) described by a vortex filament associated with a specific solutions of the Hirota equation [i.e., (3.9) for a specific choice of the triplet (A, B, C)] can be found from

$$\gamma_l(\lambda, x, t)|_{\lambda=0} \equiv \Psi^{-1}(x,\lambda;t)\frac{\partial}{\partial\lambda}\Psi(x,\lambda;t)|_{\lambda=0} = -i\sum_{i=1}^{3} x_i(x,t)\sigma_i \tag{5.1}$$

where σ_i are the Pauli matrices.

It is well known that (see [45,46] where this fact is proved in general when the flows commute) for $(x,\lambda,t) \in \mathbb{R}^3$ the (matrix) Jost solution $\Psi(x,\lambda;t)$ belongs to the unitary group $SU(2)$ and then the components $x_i(x,t)$ can be uniquely determined from (5.1).

We observe that there is no loss of generality in evaluating the left hand side of Eq. (5.1) for $\lambda = 0$. In fact, let us take η real and put

$$V^{[\eta]}(x) = e^{i\eta x\sigma_3} V(x) e^{-i\eta x\sigma_3}.$$

Then any solution $X(\lambda, x)$ of the Zakharov–Shabat system with potential $V(x)$ leads to a solution $e^{i\eta x\sigma_3}X(\lambda+\eta, x)$ of the Zakharov–Shabat system with dilated potential $V^{[\eta]}(x)$. Moreover, if $S(\lambda)$ is the original scattering matrix, then $S(\lambda+\eta)$ is the scattering matrix of the dilated ZS system. We now observe that we have shifted the entire ZS spectrum to the left by a distance of η. In other words, the identity

$$\mathrm{Tr}\,\gamma_l(\lambda = 0, x, t) \equiv 0$$

would imply $\mathrm{Tr}\,\gamma_l(\lambda, x, t) \equiv 0$ if we would apply it to a suitably dilated potential.

Now we can easily discover the differential equations whose solutions are the components of the curve described by the vortex filament. In order to get these equations, let us define

$$\mathcal{X}(\lambda, x, t) = \sum_{j=1}^{3} x_j(\lambda, x, t)\sigma_j = i\Psi(\lambda, x, t)^{-1}\Psi_\lambda(\lambda, x, t) \tag{5.2}$$

where Ψ satisfies the Zakharov–Shabat systems[2]

$$\Psi_x = [-i\lambda\sigma_3 + \tilde{V}]\Psi, \qquad \Phi_x = [-i\lambda\sigma_3 + \tilde{V}]\Phi. \tag{5.3}$$

The explicit form of Ψ is given in the last appendix (formulas (B.4) and (B.5)). In Eq. (5.3), we have (as in (2.21))

$$\tilde{V}(x) = -i\sigma_3 V(x) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} = i\left\{(\mathrm{Im}\,q)\sigma_1 + (\mathrm{Re}\,q)\sigma_2\right\}. \tag{5.4}$$

We can now easily compute (subscripts denote partial derivatives)

$$\begin{aligned}
\mathcal{X}_x &= i\left(\Psi^{-1}(\lambda, x, t)\right)_x \Psi_\lambda(\lambda, x, t) + i\Psi^{-1}(\lambda, x, t)\Psi_{\lambda x}(\lambda, x, t) \\
&= i\left[-\Psi^{-1}(\lambda, x, t)\Psi_x(\lambda, x, t)\Psi^{-1}(\lambda, x, t)\Psi_\lambda(\lambda, x, t) + \Psi^{-1}(\lambda, x, t)(\Psi_x(\lambda, x, t))_\lambda\right] \\
&= i[-\Psi^{-1}(\lambda, x, t)(-i\lambda\sigma_3 + \tilde{Q})\Psi(\lambda, x, t)\Psi^{-1}(\lambda, x, t)\Psi_\lambda(\lambda, x, t) + \Psi^{-1}(\lambda, x, t)\sigma_3\Psi(\lambda, x, t) \\
&\quad + \Psi^{-1}(\lambda, x, t)(-i\lambda\sigma_3 + \tilde{Q})\Psi_\lambda(\lambda, x, t)] = \Psi^{-1}(\lambda, x, t)\sigma_3\Psi(\lambda, x, t),
\end{aligned} \tag{5.5}$$

where $\mathcal{X}(\lambda, x, t) \sim \sigma_3 x$ as $x \to \infty$ (see (5.2)). This equation is most useful in situations where $\Psi(\lambda, x, t)$ is known, as in multisoliton cases. The basic idea is now to express the quantity $\Psi^{-1}(\lambda, x, t)\sigma_3\Psi_\lambda(\lambda, x, t)$ in terms of triplet matrices as done before for the multisoliton solutions of the Hirota equation. In particular, we use the results presented in Appendix B.

It is convenient to write the matrix $\Psi(\lambda, x, t)$ as $\Psi(\lambda, x, t) = \begin{pmatrix} \overline{\psi}^{(up)} & \psi^{(up)} \\ \overline{\psi}^{(dn)} & \psi^{(dn)} \end{pmatrix}$ where we suppressed the $(\lambda, x; t)$ dependence. Since $\Psi(\lambda, x, t)$ belongs to $SU(2)$ for $\lambda \in \mathbb{R}$, we have $\Psi^{-1}(\lambda, x, t) = \Psi^\dagger(\lambda, x, t)$ for $\lambda \in \mathbb{R}$ (we have to replace λ with λ^* in the right-hand side if $\lambda \in \mathbb{C}$) and then we get

$$\Psi^{-1}\sigma_3\Psi = \begin{pmatrix} \left|\overline{\psi}^{(up)}\right|^2 - \left|\overline{\psi}^{(dn)}\right|^2 & (\overline{\psi}^{(up)})^*\psi^{(up)} - (\overline{\psi}^{(dn)})^*\psi^{(dn)} \\ (\psi^{(up)})^*\overline{\psi}^{(up)} - (\psi^{(dn)})^*\overline{\psi}^{(dn)} & \left|\psi^{(up)}\right|^2 - \left|\psi^{(dn)}\right|^2 \end{pmatrix}. \tag{5.6}$$

The explicit form of $\Psi^{-1}\sigma_3\Psi$ can be easily obtained by means of (B.4) and (B.5).

It would be better to describe the curve (for a fixed t) described by a vortex filament associated with a specific solution of the Hirota equation in terms of its curvature and torsion. In fact, it is well known (see, for example, [47,48]) that if curvature (different from zero) and torsion are (globally) given then they characterize uniquely the curve up to a euclidean movement. As shown in [4], curvature and torsion of a vortex filament associated with a given solution of (1.2) can be easily obtained as follows:

$$\kappa = 2|q|, \qquad \tau = \frac{1}{2i}\left(\frac{q_x}{q} - \frac{q_x^*}{q^*}\right), \tag{5.7}$$

where κ and τ satisfy (1.5). Substituting in (5.7) the expression of a specific solution obtained from (3.9), we can explicitly write down curvature and torsion corresponding to the selected solution. The coefficient α_3 contributes only in exponentials but is not generically a phase, therefore it will give a contribution also to the curvature. For example, below (see Figs. 5.1 and 5.2) we plot the graphics of curvature and torsion corresponding to the double-pole solution whose triplet is the one considered in the second example of the preceding section, i.e.

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

At this point the visualization of the vortex filament corresponding to some soliton solution arising from (3.9) seems to be a natural requirement. To reach this goal, we should act as follows: (1) compute curvature and torsion as explained above, (2) integrate the Cauchy problem associated to the Frenet equations using an initial Frenet orthonormal basis $\boldsymbol{t}(-\infty)$, $\boldsymbol{n}(-\infty)$, $\boldsymbol{b}(-\infty)$, and (3) integrate the $\boldsymbol{t}$ vector to arrive at the vortex filament whose points are given by the vector $\vec{X}$ satisfying (1.4). However, also in our case where κ, τ are given in terms of elementary functions, the analytic treatment of the Cauchy problem for the Frenet equations is not easy because the

2 The Zakharov–Shabat system can be written as in (5.3) by multiplying (1.7) by $-i\sigma_3$.

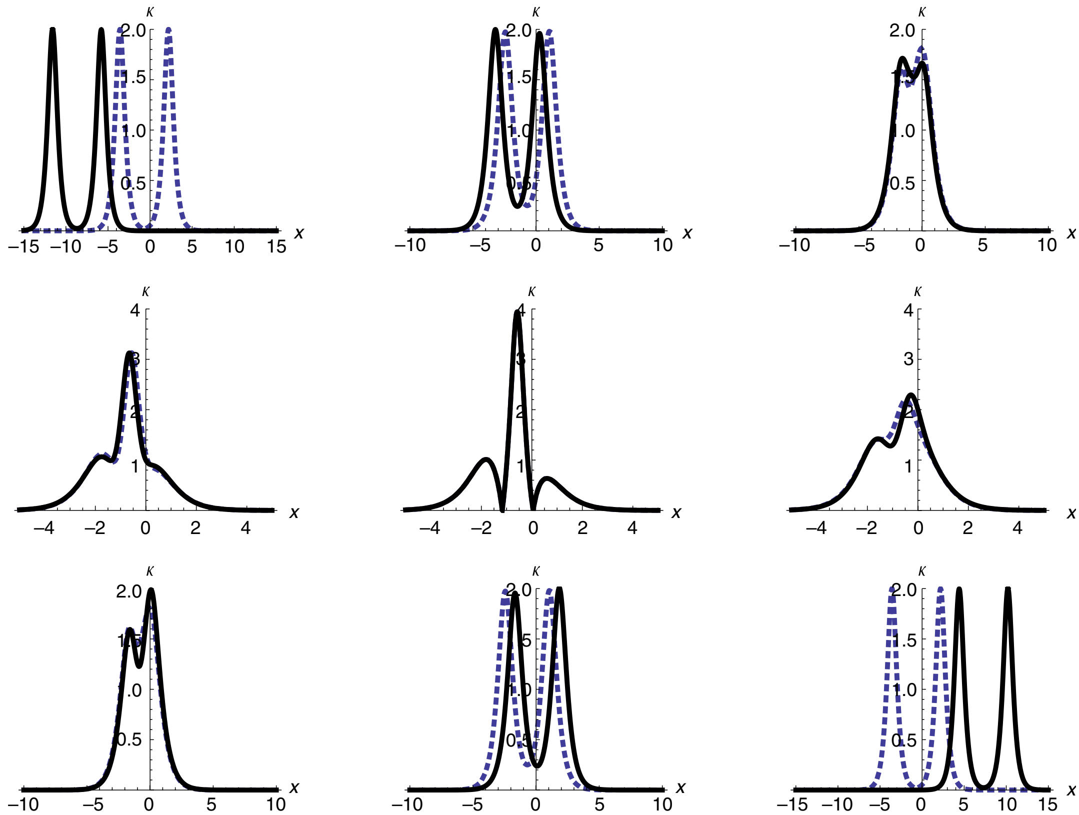

Fig. 5.1. An example of a two-poles solution in the NLS case (dotted line, $\alpha_2 = -1, \alpha_3 = 0$) versus the Hirota case (solid line, $\alpha_2 = -1, \alpha_3 = 0.1$). The matrix triplet used here is the same as in the previous section. We plot here the curvature $\kappa = |q|$. For the sake of simplicity we take an example where the torsion is not affected by the axial velocity. The time sequence is: First row $t = -20, t = -1, t = -0.3$; Second row $t = -0.1, t = 0, t = 0.2$; Third row $t = 0.3, t = 2, t = 20$. The effect of the axial velocity is to move the support of the solution in time.

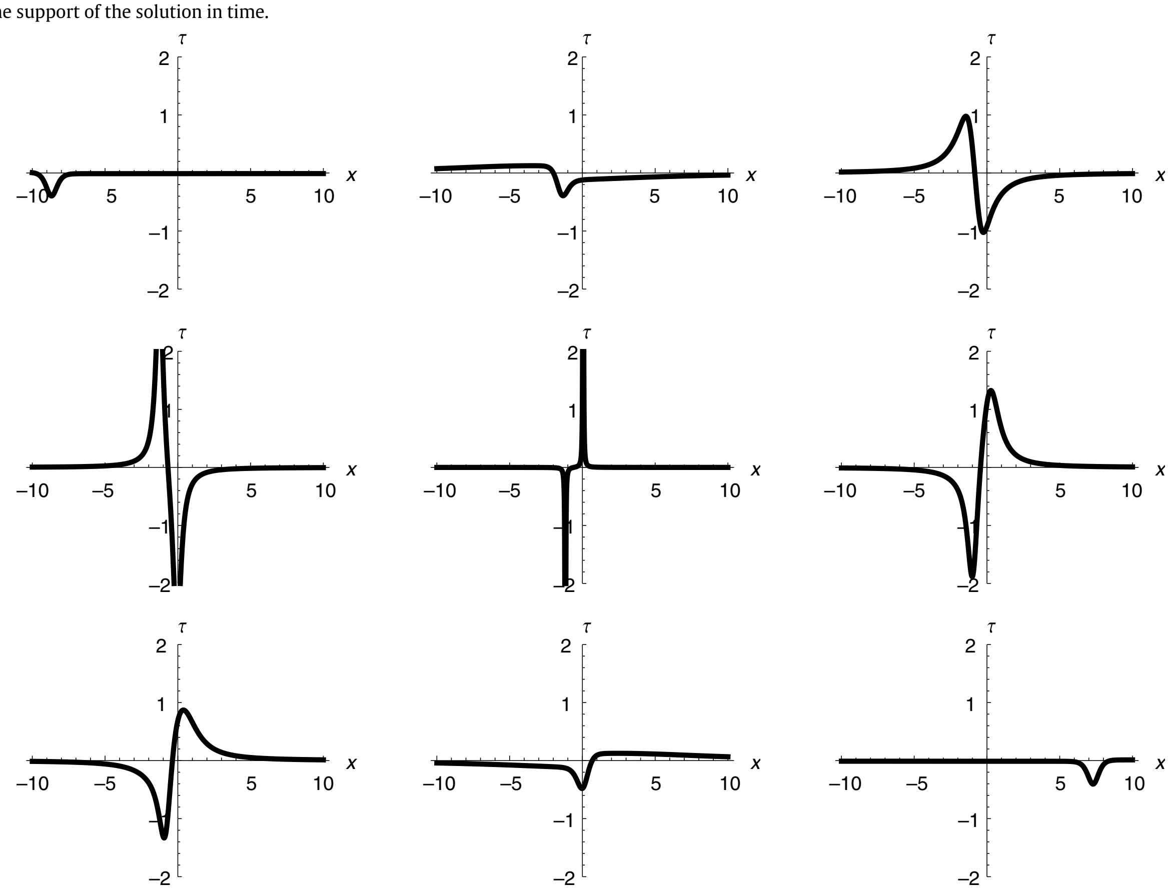

Fig. 5.2. For sake of completeness we display also the torsion evolution related to the double pole solution presented in the previous section ($\alpha_2 = -1, \alpha_3 = 0.1$). We recall that in this particular example the torsion is not affected by the axial velocity. The time sequence is: First row $t = -20, t = -2, t = -0.3$; Second row $t = -0.1, t = 0.001, t = 0.2$; Third row $t = 0.3, t = 2, t = 20$.

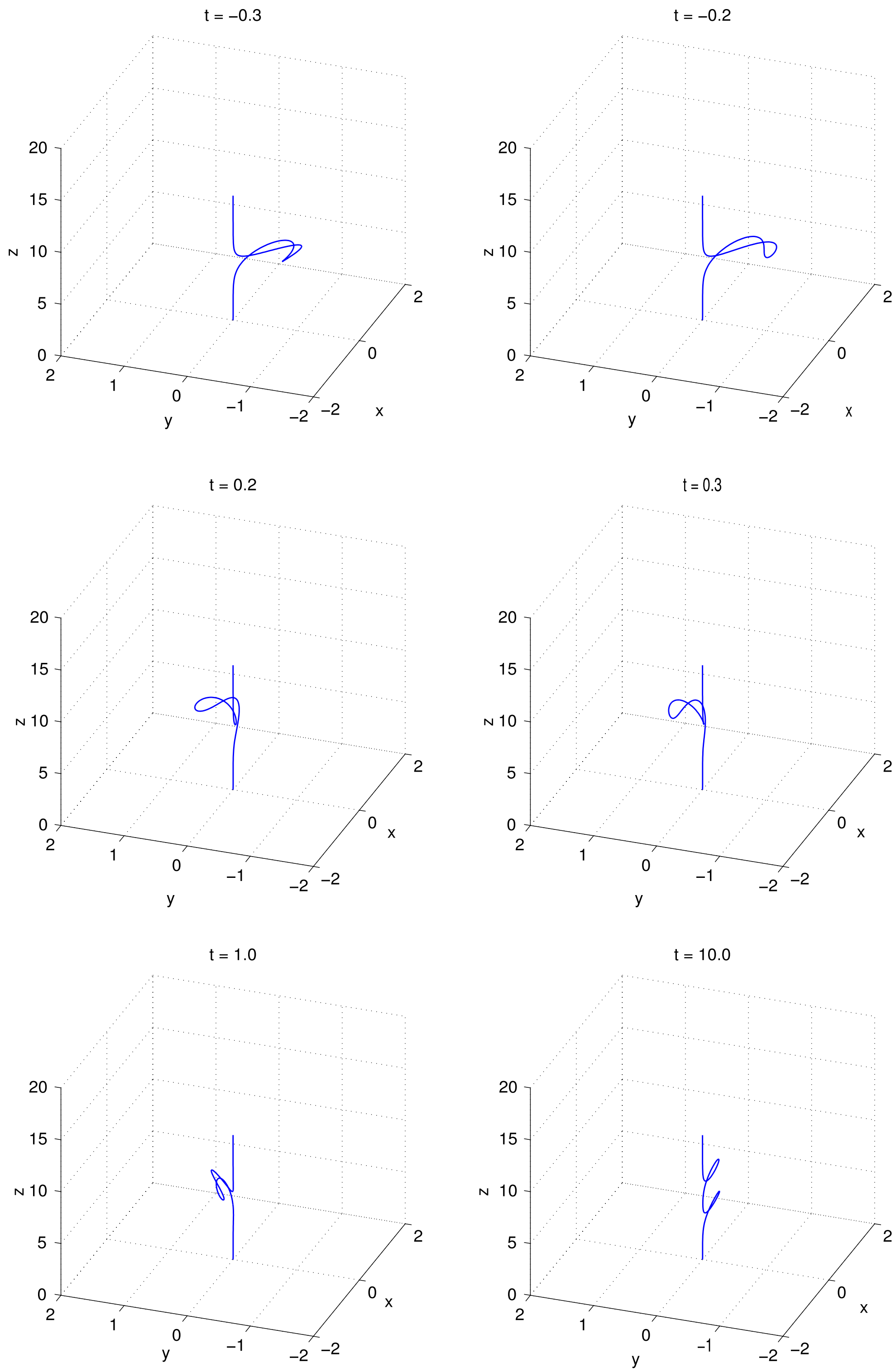

Fig. 5.3. Double pole filament motion without axial velocity ($\alpha_3 = 0$).

expressions for κ and τ (corresponding to a given solution obtained from (3.9)) is exceedingly long. We use a Matlab notebook[3] which implements the second and third steps numerically. This notebook requires κ, τ and the initial conditions $\boldsymbol{t}(-\infty)$, $\boldsymbol{n}(-\infty)$ as input and returns the plot of the curve having the given κ and τ as output. By using this software, we plot below (see Figs. 5.3 and 5.4) the curves representing the vortex filaments associated with the double-pole solution (4.5). The first series of plots given in Fig. 5.3 displays the motion of a filament with zero axial velocity, i.e. based on the pure NLS equation. The following two series of plots in 5.4 and in 5.5 represent the motion of a filament where the coefficient α_3 assumes the values $\alpha_3 = .1$ (as in the previous section) and $\alpha_3 = 1$, respectively. As can be

3 The software has been developed with the help of G. Rodriguez (University of Cagliari).

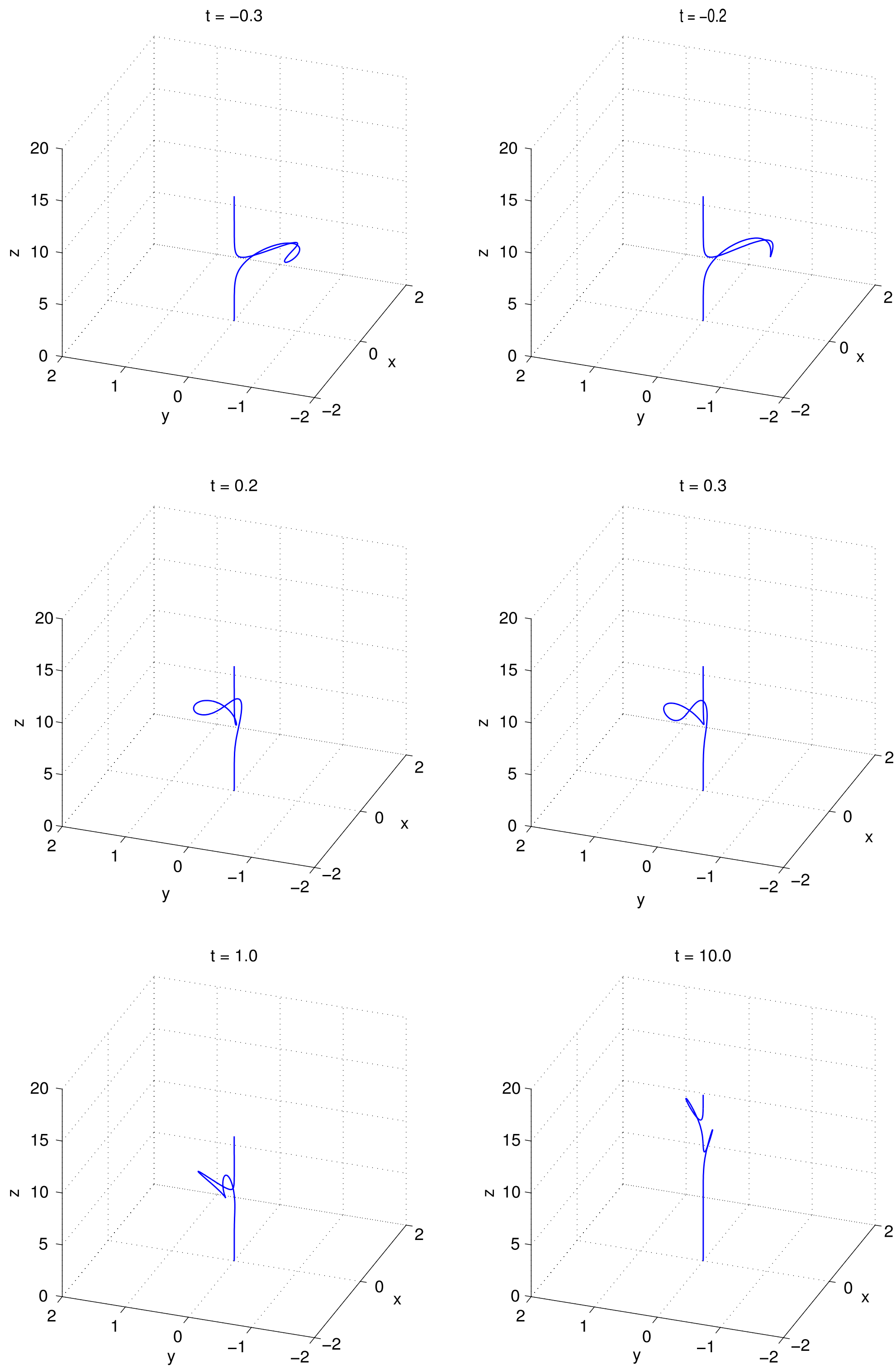

Fig. 5.4. Double pole filament motion with axial velocity $\alpha_3 = 0.1$.

observed in the formula (3.9) and remarked at the end of the previous section, the most visible effect of the axial velocity is a time shift of the support of interacting different components of the solution. Actually the curvature of the two-pole solution is

$$\kappa = \frac{8e^{8\alpha_3 t-2x}\sqrt{\left(4\alpha_2 t+\frac{1}{4}\alpha_2 t e^{16\alpha_3 t-4x}\right)^2+\left(-12\alpha_3 t-\frac{1}{16}e^{16\alpha_3 t-4x}(-12\alpha_3 t+x+1)+x\right)^2}}{e^{16\alpha_3 t-4x}\left(16{\alpha_2}^2 t^2+\left(-12\alpha_3 t+x+\frac{1}{2}\right)^2+\frac{1}{8}\right)+\frac{1}{256}e^{32\alpha_3 t-8x}+1} \tag{5.8}$$

and the same arguments applied at the end of the preceding section (for the q solution) can be applied.

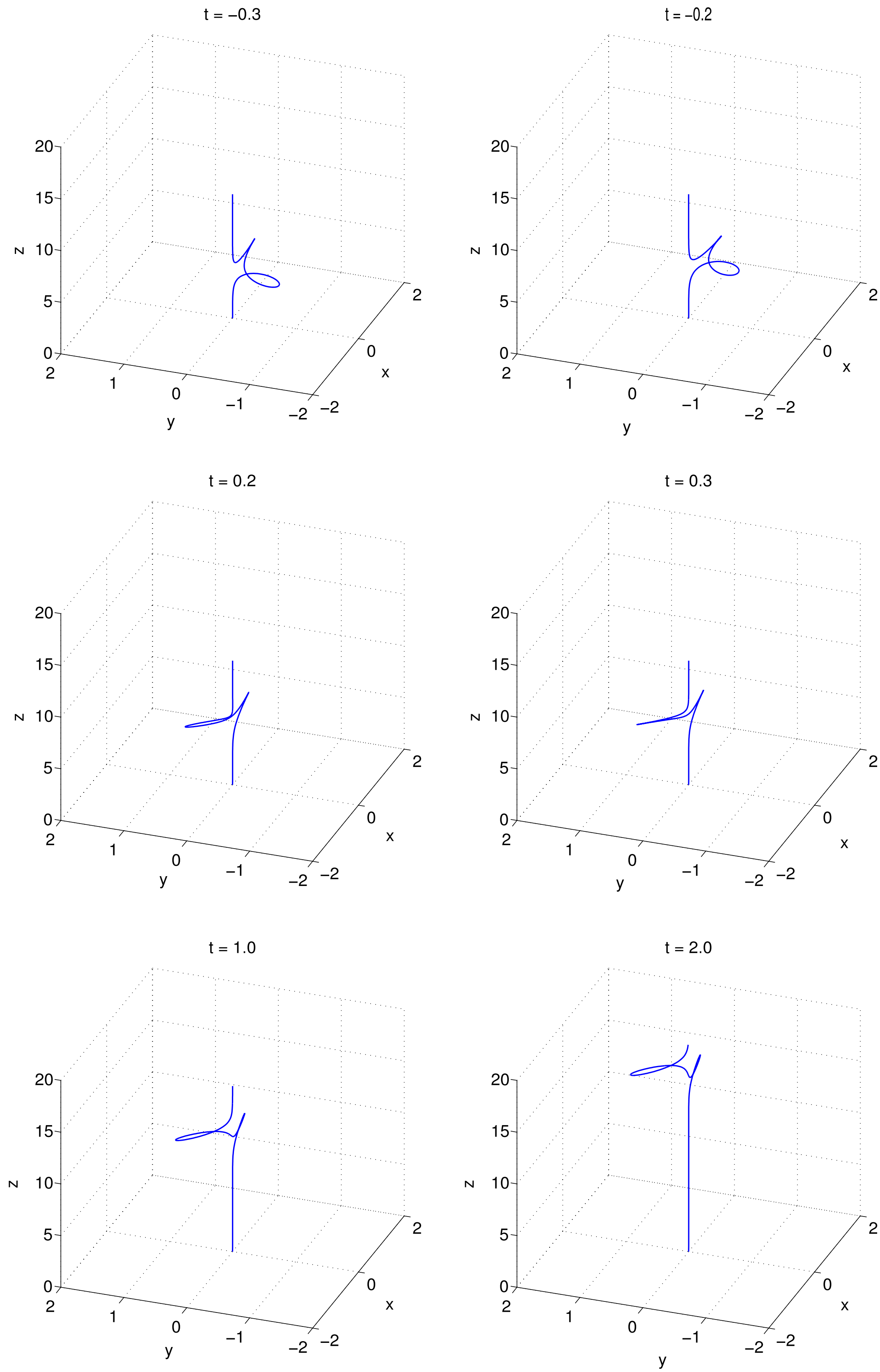

Fig. 5.5. Double pole filament motion with axial velocity $\alpha_3 = 1$. This value could be at the limit of the model where the α_3/α_2 should not be too big but it is shown here because the effects of the α_3 are more visible.

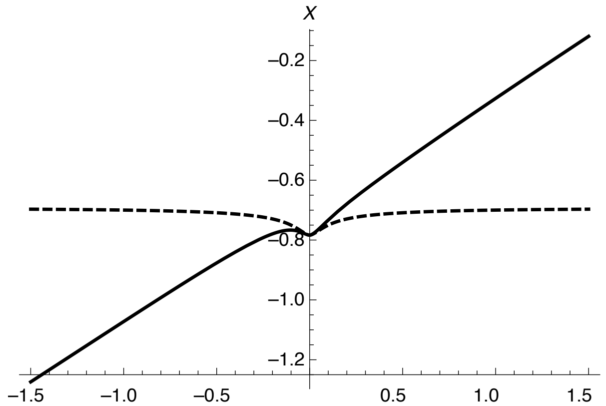

Fig. 5.6. The position at different times of the center of mass X of the curvature κ. The solid line is related to the double pole solution case with $\alpha_3 = .1$. The velocity at the interaction is nonlinear but asymptotically it reaches the predicted value $4\alpha_3 = .4$. The dashed line represents the center of mass position of the curvature for the standard NLS ($\alpha_3 = 0$) case.

The main qualitative effect of α_3 is that the center of mass of the solution can move with a velocity independent of the two peaks velocity. This effect, already remarked in the two-soliton case in [3], is true for every reflectionless solution of the Hirota equation, including the multipole solutions. This effect cannot be reabsorbed by a suitable Galilean boost because the interaction structure is quantitatively different. In Fig. 5.6 we plot the center of mass of the curvature related to the double pole solutions in the Hirota and NLS cases.

Finally we remark that from the purely mathematical point of view a suitable curve motion can be reconstructed for every value of the ratio α_3/α_2. However, as remarked in [3], α_3 is a small parameter in the LIA development and therefore it makes sense in the fluid-dynamics context only for not too large ratios α_3/α_2.

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Appendix A. Independent proof of formula (3.9)

We prove that the function $q(x, t)$ expressed by (3.9) satisfies Eq. (1.2) by simply computing the quantities $q_t,\ q_x,\ q_{xx},\ q_{xxx},\ 6|q|^2q_x$, $2|q|^2q$ and substituting their expressions in (3.9). Similar calculations have already been done in [24] for the NLS equation, and in [25] for the cmKdV. We need some preliminary definitions and results.

Definition A.1. Let us define $\hat{\Gamma}(x; t) = \Gamma(x; t)^\dagger = I_p + P(x)\overline{P}(x; t)$ (where $\overline{P}(x, t)$, $P(x)$ and $\Gamma(x; t)$ are given by Eqs. (3.10), (3.11), (3.12), respectively) and

$$\boldsymbol{X}_n = (A^\dagger)^n + (-1)^n\overline{P}(x, t)A^nP(x). \tag{A.1}$$

From now on, for the sake of convenience we disregard the dependence of $q,\ \Gamma, \overline{P}, P$ on (x, t). We have

Lemma A.2.

$$(A^\dagger\overline{P} + \overline{P}A)\hat{\Gamma}^{-1}(AP + PA^\dagger) = \boldsymbol{X}_2 - \boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1. \tag{A.2}$$

Proof. Using that (see [24] for details on these formulas)

$$\overline{P}\hat{\Gamma}^{-1} = \Gamma^{-1}\overline{P}, \qquad \hat{\Gamma}^{-1}P = P\Gamma^{-1},$$
$$\overline{P}P\Gamma^{-1} = \Gamma^{-1}\overline{P}P = I_p - \Gamma^{-1}, \qquad \hat{\Gamma}^{-1} = I_p - P\Gamma^{-1}\overline{P},$$

we get

$$\begin{aligned}(A^\dagger\overline{P} + \overline{P}A)\hat{\Gamma}^{-1}(AP + PA^\dagger) &= A^\dagger\Gamma^{-1}\overline{P}AP + \overline{P}AP\Gamma^{-1}A^\dagger + A^\dagger(I - \Gamma^{-1})A^\dagger + \overline{P}A\hat{\Gamma}^{-1}AP \\ &= [(A^\dagger)^2 + \overline{P}A^2P] - (A^\dagger - \overline{P}AP)\Gamma^{-1}(A^\dagger - \overline{P}AP) \\ &= \boldsymbol{X}_2 - \boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1,\end{aligned}$$

which completes the proof. ■

Lemma A.3. *The time derivative of $q(x,t)$ is given by*

$$q_t = -2iB^\dagger e^{-xA^\dagger}\Gamma^{-1}\left[-4\alpha_2\boldsymbol{X}_2 - 8i\alpha_3\boldsymbol{X}_3\right]\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger. \tag{A.3}$$

Proof. A direct computation gives us

$$q_t = 2B^\dagger e^{-xA^\dagger}\Gamma^{-1}\frac{\partial\Gamma}{\partial t}\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger - 2B^\dagger e^{-xA^\dagger}\Gamma^{-1}e^{-xA^\dagger}(i\phi(-iA^\dagger))e^{i\phi(-iA^\dagger)t}C^\dagger.$$

Taking into account that

$$\frac{\partial\Gamma}{\partial t} = i\phi(-iA^\dagger)\left(\Gamma - I_p\right) + \overline{P}(-i\phi(iA))P,$$
$$\phi(-iA^\dagger) + i\overline{P}\left(-i\phi(iA)\right)P = -4\alpha_2\boldsymbol{X}_2 - 8i\alpha_3\boldsymbol{X}_3,$$

Eq. (A.3) is easily obtained. ■

Lemma A.4. *We have*

$$q_x = 4B^\dagger e^{-xA^\dagger}\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger, \tag{A.4}$$
$$q_{xx} = -8B^\dagger e^{-xA^\dagger}\Gamma^{-1}\left(2\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_2\right)\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger, \tag{A.5}$$
$$q_{xxx} = 16B^\dagger e^{xA^\dagger}\Gamma^{-1}\Big(6\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1 - 3\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_2 - 3\boldsymbol{X}_2\Gamma^{-1}\boldsymbol{X}_1 + \boldsymbol{X}_3\Big)\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger. \tag{A.6}$$

Proof. By using the formula $\frac{\partial\Gamma^{-1}}{\partial x} = -\Gamma^{-1}\frac{\partial\Gamma}{\partial x}\Gamma^{-1}$ and Eqs. (3.13) and (3.14), we easily get

$$\left(e^{-xA^\dagger}\Gamma^{-1}e^{-xA^\dagger}\right)_x = -2e^{-xA^\dagger}\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}e^{-xA^\dagger}, \tag{A.7}$$
$$\left(e^{xA^\dagger}\boldsymbol{X}_1e^{xA^\dagger}\right)_x = 2e^{xA^\dagger}\boldsymbol{X}_2e^{xA^\dagger}, \tag{A.8}$$
$$\left(e^{xA^\dagger}\boldsymbol{X}_2e^{xA^\dagger}\right)_x = 2e^{xA^\dagger}\boldsymbol{X}_3e^{xA^\dagger}. \tag{A.9}$$

To calculate q_x we write

$$q_x = -2B^\dagger\left(e^{-xA^\dagger}\Gamma^{-1}e^{-xA^\dagger}\right)_x e^{i\phi(-iA^\dagger)t}C^\dagger,$$

and applying formula (A.7) we obtain Eq. (A.4). To prove Eq. (A.5), it is enough to observe that

$$q_{xx} = 4B^\dagger\left(e^{-xA^\dagger}\Gamma^{-1}e^{-xA^\dagger}e^{xA^\dagger}\boldsymbol{X}_1e^{xA^\dagger}e^{-xA^\dagger}\Gamma^{-1}e^{-xA^\dagger}\right)_x e^{i\phi(-iA^\dagger)t}C^\dagger$$

and by using formula (A.7) (twice) and formula (A.8), we arrive at Eq. (A.5). Finally, since

$$q_{xxx} = -16B^\dagger\left(e^{-xA^\dagger}\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}e^{-xA^\dagger}\right)_x e^{i\phi(-iA^\dagger)t}C^\dagger + 8B^\dagger\left(e^{-xA^\dagger}\Gamma^{-1}\boldsymbol{X}_2\Gamma^{-1}e^{-xA^\dagger}\right)_x e^{i\phi(-iA^\dagger)t}C^\dagger$$

with the help of (A.7), (A.8) and (A.9), it is immediate to prove (A.5). ■

Lemma A.5. *The following identities hold*

$$2|q|^2q = -8B^\dagger e^{-xA^\dagger}\Gamma^{-1}\left(-2\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1 + 2\boldsymbol{X}_2\right)\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger, \tag{A.10}$$
$$6|q|^2q_x = 16B^\dagger e^{-xA^\dagger}\Gamma^{-1}\left(-6\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_1 + 3\boldsymbol{X}_1\Gamma^{-1}\boldsymbol{X}_2 + 3\boldsymbol{X}_2\Gamma^{-1}\boldsymbol{X}_1\right)\Gamma^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger. \tag{A.11}$$

Proof. To prove formula (A.10) we, first of all, observe that $2|q|^2q = 2qq^\dagger q$. Substituting (3.9) in the right hand side of the preceding equation and taking into account formulas (3.13), (3.14) and (A.2) we easily derive (A.10). The proof of (A.11) proceeds in a similar direct way after we have written $6|q|^2q_x = 3qq^\dagger q_x + 3q_xq^\dagger q$. ■

We are ready to establish the following.

Theorem A.6. *Given a triplet of matrices (A,B,C) in the admissible class, the function*

$$q(x,t) = -2B^\dagger e^{-A^\dagger x}\Gamma^{-1}(x,t)e^{-A^\dagger x+i\phi(-iA^\dagger)t}C^\dagger, \tag{A.12}$$

satisfies the Hirota equation (1.2). Moreover, this solution is globally defined in the xt-plane and decays exponentially as $x\to\pm\infty$ for each fixed t.

Proof. Substituting the right hand side of Eqs. (A.3), (A.5), (A.6), (A.10), (A.11) in the Hirota equation (1.2), we get $0 = 0$. The properties of our function follow from Propositions 3.2 and 3.4. ■

Appendix B. Expressing the Jost solutions in terms of the triplet matrices

In this section we obtain the explicit expressions for the matrix $\Psi(x,\lambda)$ and its inverse $\Psi^\dagger(x,\lambda)$ in terms of the matrix triplet introduced in Section 3 to solve the Marchenko equations. To do so, we start by recalling the notation introduced

$$\alpha_l(x,y) = \begin{pmatrix}\overline{K}(x,y) & K(x,y)\end{pmatrix} = \begin{pmatrix}\overline{K}^{(up)}(x,y) & K^{(up)}(x,y)\\ \overline{K}^{(dn)}(x,y) & K^{(dn)}(x,y)\end{pmatrix},$$

and write the Marchenko equations (2.10a) in scalar form as follows:

$$\overline{K}(x,y) + \begin{pmatrix}0\\1\end{pmatrix}\Omega_l(x+y) + \int_x^\infty dz\, K(x,z)\Omega_l(z+y) = 0_{2\times1}, \tag{B.1a}$$

$$K(x,y) + \begin{pmatrix}1\\0\end{pmatrix}\breve{\Omega}_l(x+y) + \int_x^\infty dz\, \overline{K}(x,z)\breve{\Omega}_l(z+y) = 0_{2\times1}, \tag{B.1b}$$

where $\Omega_l(y) = Ce^{-xA}e^{-i\phi(iA)t}B$ and $\breve{\Omega}_l(y) = -B^\dagger e^{-yA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger$.

By following the procedure shown in Section 3, we easily get the solutions of the Marchenko equations (B.1) and they read as follows:

$$K^{(up)}(x,y;t) = B^\dagger e^{-xA^\dagger}\Gamma^{-1}(x;t)e^{-yA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger, \tag{B.2a}$$

$$K^{(dn)}(x,y;t) = -Ce^{-xA}P(x)\Gamma^{-1}(x;t)e^{-yA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger, \tag{B.2b}$$

$$\overline{K}^{(up)}(x,y;t) = -B^\dagger e^{-xA^\dagger}\overline{P}(x;t)(\Gamma^{-1}(x;t))^\dagger e^{-yA}e^{-i\phi(iA)t}B, \tag{B.2c}$$

$$\overline{K}^{(dn)}(x,y;t) = -Ce^{-xA}(\Gamma^{-1}(x;t))^\dagger e^{-yA}e^{-i\phi(iA)t}B, \tag{B.2d}$$

where Γ, P and $\overline{P}$ have been introduced in (3.12), (3.10) and (3.11) while Q and N have been introduced in (3.6). The relationship between the functions $K^{(up)}(x,y)$, $\overline{K}^{(up)}(x,y)$, $K^{(dn)}(x,y)$ and $\overline{K}^{(dn)}(x,y)$ and the Jost solutions are given by (2.9a) and (2.9b) which can be written as

$$\overline{\psi}^{(up)}(\lambda,x;t) = e^{-i\lambda x} + \int_x^\infty dy\overline{K}^{(up)}(x,y;t)e^{-i\lambda y}, \tag{B.3a}$$

$$\overline{\psi}^{(dn)}(\lambda,x;t) = \int_x^\infty dy\overline{K}^{(dn)}(x,y;t)e^{-i\lambda y}, \tag{B.3b}$$

$$\psi^{(up)}(\lambda,x;t) = \int_x^\infty dyK^{(up)}(x,y;t)e^{i\lambda y}, \tag{B.3c}$$

$$\psi^{(up)}(\lambda,x;t) = e^{i\lambda x} + \int_x^\infty dyK^{(dn)}(x,y;t)e^{i\lambda y}. \tag{B.3d}$$

Substituting (B.2) into (B.3) we have

$$\overline{\psi}^{(up)}(\lambda,x;t) = e^{-i\lambda x}\left[1 + iB^\dagger e^{-xA^\dagger}\overline{P}(x)(\Gamma^\dagger(x))^{-1}(\lambda I_p - iA)^{-1}e^{-xA}e^{-i\phi(iA)t}B\right], \tag{B.4a}$$

$$\overline{\psi}^{(dn)}(\lambda,x;t) = e^{-i\lambda x}\left[iCe^{-xA}(\Gamma^\dagger(x))^{-1}(\lambda I_p - iA)^{-1}e^{-xA}e^{-i\phi(iA)t}B\right], \tag{B.4b}$$

$$\psi^{(up)}(\lambda,x;t) = e^{i\lambda x}\left[iB^\dagger e^{-xA^\dagger}\Gamma^{-1}(x)(\lambda I_p + iA^\dagger)^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger\right], \tag{B.4c}$$

$$\psi^{(dn)}(\lambda,x;t) = e^{i\lambda x}\left[1 - iCe^{-xA}P(x)\Gamma^{-1}(x)(\lambda I_p + iA^\dagger)^{-1}e^{-xA^\dagger}e^{i\phi(-iA^\dagger)t}C^\dagger\right], \tag{B.4d}$$

and easy calculations also give us

$$(\overline{\psi}^{(up)}(\lambda,x))^\dagger = e^{i\lambda x}\left[1 - iB^\dagger e^{i\phi(-iA^\dagger)t}e^{-xA^\dagger}(\lambda I_p + iA^\dagger)^{-1}\Gamma^{-1}(x)\overline{P}(x)e^{-xA}B\right], \tag{B.5a}$$

$$(\overline{\psi}^{(dn)}(\lambda,x))^\dagger = e^{i\lambda x}\left[-iB^\dagger e^{i\phi(-iA^\dagger)t}e^{-xA^\dagger}(\lambda I_p + iA^\dagger)^{-1}\Gamma^{-1}(x)e^{-xA^\dagger}C^\dagger\right], \tag{B.5b}$$

$$(\psi^{(up)}(\lambda,x))^\dagger = e^{-i\lambda x}\left[-iCe^{-i\phi(iA)t}e^{-xA}(\lambda I_p - iA)^{-1}(\Gamma^\dagger(x))^{-1}e^{-xA}B\right], \tag{B.5c}$$

$$(\psi^{(dn)}(\lambda,x))^\dagger = e^{-i\lambda x}\left[1 + iCe^{-i\phi(iA)t}e^{-xA}(\lambda I_p - iA)^{-1}(\Gamma^\dagger(x))^{-1}P(x)e^{-xA^\dagger}C^\dagger\right]. \tag{B.5d}$$

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