degrees of freedom. If we add to the n variables u_k the m quantities λ_i as additional variables and ask for the stationary value of the function $\overline{F}$, this variation problem gives the same n equations as we had before if we vary with respect to the u_k, while the variations of the λ_i give the m additional conditions

$$f_1 = 0, \ldots, f_m = 0. \tag{25.24}$$

These are exactly the given auxiliary conditions, but now obtained *a posteriori*, on account of the variation problem.

The method of Lagrange permits the use of surplus coordinates—a great convenience in many considerations of mechanics. It preserves the full symmetry of all coordinates by making it unnecessary to distinguish between dependent and independent variables.

> *Summary.* The Lagrangian-multiplier method reduces a variation problem with auxiliary conditions to a free variation problem without auxiliary conditions. We modify the given function F, which is to be made stationary, by adding the left-hand sides of the auxiliary conditions, after multiplying each by an undetermined factor λ. Then we handle the modified problem as a free variation problem. The resulting conditions, together with the given auxiliary conditions, determine the unknowns and the λ-factors.

6. Non-holonomic auxiliary conditions. As was pointed out in chap. I, section 6, the restrictions on the mechanical variables of a problem may be given in a differential instead of a finite form. We then have a variation problem with non-holonomic auxiliary conditions. The equations (25.13) do not exist in this case, but we have relations analogous to the *differentiated* forms (25.14) of the auxiliary conditions. The only difference is that the left-hand sides of these equations are no longer exact differentials but merely infinitesimal quantities. We can write the non-holonomic conditions in the following form:

$$\begin{aligned} \overline{\delta f}_1 &= A_{11}\delta u_1 + A_{12}\delta u_2 + \ldots + A_{1n}\delta u_n = 0, \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \overline{\delta f}_m &= A_{m1}\delta u_1 + A_{m2}\delta u_2 + \ldots + A_{mn}\delta u_n = 0. \end{aligned} \tag{26.1}$$

Here the A_{ik} are given functions of the u_i which cannot be considered as the partial derivatives of a function f_i.

Non-holonomic conditions cannot be handled by the elimination method, because the equations for eliminating some variables as dependent variables do not exist. The Lagrangian λ-method, however, is again available. By exactly the same procedure as before, we can obtain an equation analogous to (25.20), namely:

$$\delta F + \lambda_1 \overline{\delta f}_1 + \ldots + \lambda_m \overline{\delta f}_m = 0 \; ; \tag{26.2}$$

and again all the δu_k are handled as free variations. The only difference lies in the fact that we cannot proceed to the equation (25.21) and have to be content with the differential formulation of the procedure. The reduction of a conditioned variation problem to a free variation problem is once more accomplished.

> *Summary.* The Lagrangian λ-method is applicable even to non-holonomic conditions. We multiply the left sides of these conditions by some undetermined λ-factors and add them to the variation of the function F which is to be made stationary. This whole expression is put equal to zero, considering all the variations δu_k as free variations.

7. The stationary value of a definite integral. The analytical problems of motion involve a special type of extremum problem: the stationary value of a *definite integral*. The branch of mathematics dealing with problems of this nature is called the Calculus of Variations. A typical problem of this kind is that of the brachistochrone (the curve of quickest descent), first formulated and solved by John Bernoulli (1696); it is one of the earliest instances of a variational problem. We wish to find a suitable plane curve along which a particle descends in the shortest possible time, starting from A and arriving at B. If

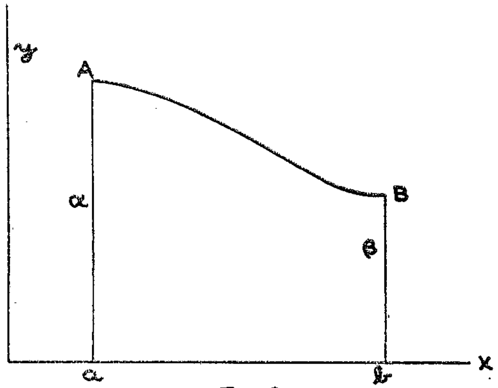

FIG. 2

the unknown curve is analytically characterized by the equation

$$y = f(x), \tag{27.1}$$

the time which must be minimized, is given by the following definite integral:

$$t = \frac{1}{\sqrt{2g}} \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{\alpha - y}}\, dx. \tag{27.2}$$

Here y is an unknown function of x which is to be determined. Amongst all possible $y = f(x)$ we want to find that particular $f(x)$ which yields the smallest possible value of t. The general conditions are that y must be a continuous and differentiable function of x, with a continuous tangent. Moreover, since the two end points of the unknown curve $y = f(x)$ are given, y has to satisfy the following boundary conditions:

$$f(a) = \alpha, \quad f(b) = \beta. \tag{27.3}$$

There is one and only one function $f(x)$ which satisfies these conditions.

The type of problem we encounter here can be more generally characterized as follows: we are given a function F of three variables:

$$F = F(y, y', x) \tag{27.4}$$

(in the above example F happens to be independent of x, but it is unnecessary to make that restriction); and given the definite integral

$$I = \int_a^b F(y, y', x)dx; \tag{27.5}$$

we are given also the boundary conditions

$$f(a) = \alpha, \quad f(b) = \beta. \tag{27.6}$$

The problem is to find a function

$$y = f(x) \tag{27.7}$$

—restricted by the customary regularity conditions—which will make the integral I an extremum, or at least give it a stationary value.

At first sight this problem appears utterly different from the previous problem where we dealt with the extremum or stationary value of a function $F(u_1, \ldots, u_n)$ of a set of variables. Instead of a *function*, a *definite integral* must be minimized. Moreover, instead of a *set of variables* $u_1, \ldots, u_n$ we have a certain *unknown function* $y = f(x)$ at our disposal. Yet closer inspection reveals that the mathematical nature of this new problem is not substantially different from that of the previous problem.

Let us make use of the concept of the "function space," originated by Hilbert. The arbitrary function $y = f(x)$—provided that it satisfies certain general continuity requirements—can be expanded in an infinite Fourier series throughout the given range between a and b:

$$\begin{aligned} f(x) = \tfrac{1}{2} a_0 + a_1 \cos \xi + a_2 \cos 2\xi + \ldots \\ + b_1 \sin \xi + b_2 \sin 2\xi + \ldots, \end{aligned} \tag{27.8}$$

with

$$\xi = \frac{2\pi}{b - a}\left(x - \frac{a + b}{2}\right).$$

The coefficients of this expansion are uniquely determined. Then we can associate a definite set of coefficients $a_0, a_1, \ldots, a_n; b_1, \ldots, b_n$ with any function $y = f(x)$, provided we choose n large enough to make the remainder of the expansion sufficiently small. We now plot these coefficients as rectangular coordinates of a point P in a $(2n + 1)$-dimensional space. In this picture an arbitrary function is mapped as a definite point of a space of many dimensions; the value of the integral I associated with this $f(x)$ can be plotted perpendicularly as an additional dimension. We thus return to the picture of a *surface in a many-dimensional space*. A small change of the function $f(x)$

means a small variation of the position of the point P. The problem of finding the function $f(x)$ which minimizes the definite integral I is translated into the problem of finding the coordinates of the deepest point of a surface in a space of $2n+2$ dimensions. This is exactly the problem we have studied in the previous sections of this chapter.

Euler has shown how the problem can be solved by elementary means, without resorting to the tools of a specific calculus. We make use of the fact that a definite integral can be replaced by a sum of an increasing number of terms. Moreover, the derivative can be replaced by a difference coefficient. The errors we commit in this procedure can be made as small as we wish.

In accordance with the customary procedure of the calculus, we divide the interval between $x = a$ and $x = b$ into many equal small intervals, obtaining a set of abscissa values:

$$x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_n,\ x_{n+1} = b, \tag{27.9}$$

and the corresponding ordinates

$$y_0 = \alpha,\ y_1,\ y_2,\ \ldots,\ y_n,\ y_{n+1} = \beta, \tag{27.10}$$

where

$$y_k = f(x_k). \tag{27.11}$$

We then replace the derivative $f'(x_k)$ by the difference coefficient

$$z_k = \left(\frac{\Delta y}{\Delta x}\right)_{x=x_k} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}, \tag{27.12}$$

and the integral (27.5) by the sum

$$S = \sum_{k=0}^{n} F(y_k, z_k, x_k)\ (x_{k+1} - x_k). \tag{27.13}$$

This involves a certain error which, however, tends toward zero as each one of the intervals $\Delta x_k = x_{k+1} - x_k$ tends toward zero.

We replace the original integral by the sum (27.13) and ask for the stationary value of this sum. This new problem is of the customary type. We have a given function S of the n variables $y_1, \ldots, y_n$ which take the place of our earlier $u_1, \ldots, u_n$. We know that this problem is solved by setting the partial derivatives of S with respect to y_k equal to zero. Then we investigate what happens to these conditions as Δx_k tends toward zero.

Before carrying out this program we alter the expression (27.13) to an extent which becomes negligible in the limit. Since

y_k and y_{k+1} are arbitrarily near to each other, it is permissible to change $F(y_k, z_k, x_k)$ to $F(y_{k+1}, z_k, x_k)$. Hence the function we want to minimize is finally defined as follows:

$$S' = \sum_{j=0}^{n} F(y_{j+1}, z_j, x_j)(x_{j+1} - x_j). \tag{27.14}$$

Now, in forming the partial derivative of S' with respect to, say, y_{k+1}, we have to bear in mind that y_{k+1} appears in the sum S' in *two* neighbouring terms, namely, those for which $j = k$ and $j = k+1$, in view of the definition (27.12) of z_{k+1}. Partial differentiation with respect to y_{k+1} gives therefore

$$\frac{\partial S'}{\partial y_{k+1}} = \left(\frac{\partial F}{\partial y}\right)_{x=x_k} (x_{k+1} - x_k) + \left(\frac{\partial F}{\partial y'}\right)_{x=x_k} - \left(\frac{\partial F}{\partial y'}\right)_{x=x_{k+1}} \tag{27.15}$$

Dividing by $\Delta x_k = x_{k+1} - x_k$, this equation can be written as follows:

$$\left[\frac{\partial F}{\partial y} - \frac{\Delta}{\Delta x}\left(\frac{\partial F}{\partial y'}\right)\right]_{x=x_k} = 0,\ (k = 0, 1, 2, \ldots, n-1). \tag{27.16}$$

Here we have the necessary, and also sufficient conditions for the sum S' to be stationary. It is important to note that the two limiting ordinates y_0 and y_{n+1} are *given quantities* and thus remain unvaried. Should they also be unknown, we would get two boundary conditions in addition to the equations (27.16).

In the limit, when Δx decreases toward zero, the difference equation (27.16) becomes a *differential equation*. Moreover, since the points x_k come arbitrarily near to any point of the interval (a, b), that differential equation has to hold for the entire interval:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,\ (a \leqq x \leqq b). \tag{27.17}$$

This fundamental equation was discovered independently by Euler and Lagrange and is usually called the Euler-Lagrange differential equation. We notice that this differential equation is derivable by elementary means, as the condition that the sum which replaces the integral originally given shall be stationary.

This method of deriving the basic differential equation of the calculus of variations, essentially due to Euler, is objectionable from the rigorous point of view since it makes use of a double limit process in a fashion which is not necessarily admissible. The direct method of Lagrange which we shall discuss in the next chapter is free from this objection.

Summary. The problem of minimizing a definite integral which contains an unknown function and its derivative can be reduced to the elementary problem of minimizing a function of many variables. For this purpose the integral is replaced by a sum and the derivative by a difference coefficient. The condition for the vanishing of the first variation takes the form of a difference equation which in the limit goes over into the differential equation of Euler and Lagrange.

8. The fundamental processes of the calculus of variations. Lagrange realized that the problem of minimizing a definite integral requires specific tools, different from those of the ordinary calculus. With the help of these tools we can attack the problem directly, instead of reverting to the limiting process by which Euler obtained the solution of the problem.

Consider the function $y = f(x)$ which by hypothesis gives a stationary value to the definite integral (27.5). In order to prove that we *do* have a stationary value, we have to evaluate the same integral for a slightly *modified* function $y = \overline{f(x)}$ and show that the rate of change of the integral due to the change in the function becomes zero.

Now the modified function $\overline{f(x)}$ can obviously be written in the form

$$\overline{f(x)} = f(x) + \epsilon\phi(x), \tag{28.1}$$

where $\phi(x)$ is some arbitrary new function which satisfies the same general continuity conditions as $\overline{f(x)}$. Hence $\phi(x)$ has to be continuous and differentiable.

It is obvious that the selection of the hypothetical function $f(x)$ which solves our variation problem has to be made from the class of continuous

functions which can be differentiated at least once, because otherwise the integral $F(y, y', x)$ would have no meaning. However, we want to restrict our $f(x)$ by the further condition that even $f''(x)$ exists throughout the range. (A similar assumption for $\phi(x)$ is not required.)

Making use of the variable parameter ϵ we have it in our power to modify the function $f(x)$ by arbitrarily small amounts. For that purpose we let ϵ decrease toward zero.

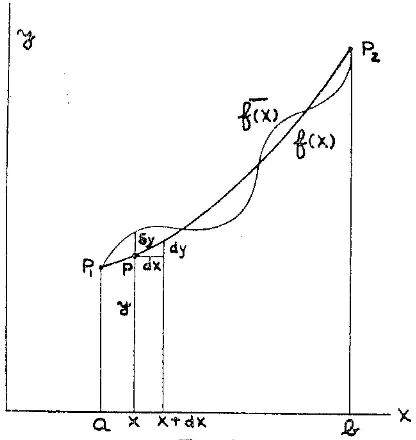

FIG. 3

We now compare the values of the modified function $\overline{f(x)}$ with the values of the original function $f(x)$ *at a certain definite point* x of the independent variable, by forming the difference between $\overline{f(x)}$ and $f(x)$. This difference is called the "variation" of the function $f(x)$ and is denoted by δy:

$$\delta y = \overline{f(x)} - f(x) = \epsilon\phi(x). \tag{28.2}$$

The variation of a function—in full analogy with the variation of the position of a point, studied before—is characterized by two fundamental features. It is an *infinitesimal* change, since the parameter ϵ decreases toward zero. Moreover, it is a *virtual* change, that means we make it in any arbitrary manner we please. Hence $\phi(x)$ is a function chosen arbitrarily, so long as the general continuity conditions are satisfied.

Note the fundamental difference between δy and dy. Both are infinitesimal changes of the function y. However the dy

refers to the infinitesimal change of the given function $f(x)$ caused by the infinitesimal change dx of the independent variable, while δy is an infinitesimal change of y which produces a *new function* $y + \delta y$.

It is in the nature of the process of variation that only the dependent function y should be varied while the variation of x serves no useful purposes. Hence we agree that we shall always put

$$\delta x = 0. \tag{28.3}$$

Moreover, if the two limiting ordinates $f(a)$ and $f(b)$ of the function $f(x)$ are prescribed, these two ordinates cannot be varied, which means

$$\begin{aligned} [\delta f(x)]_{x=a} &= 0, \\ [\delta f(x)]_{x=b} &= 0. \end{aligned} \tag{28.4}$$

We then speak of a "variation between definite limits."

> *Summary.* The calculus of variations considers a virtual infinitesimal change of a function $y = f(x)$. The variation δy refers to an arbitrary infinitesimal change of the value of y, *at the point* x. The independent variable x does not participate in the process of variation.

9. The commutative properties of the δ-process. The variation of the function $f(x)$ defines an entirely new function $\epsilon\phi(x)$. We can take the derivative of this function. On the other hand, we can take the derivative of the new function $\overline{f(x)}$ and the derivative of the original function $f(x)$. The difference of these two derivatives can naturally be called the variation of the derivative $f'(x)$. In the first case we have "the derivative of the variation":

$$\frac{d}{dx}\delta y = \frac{d}{dx}\left[\overline{f(x)} - f(x)\right] = \frac{d}{dx}\epsilon\phi(x) = \epsilon\phi'(x). \tag{29.1}$$

In the second case we have "the variation of the derivative":

$$\delta\frac{d}{dx}f(x) = \overline{f'(x)} - f'(x) = (y' + \epsilon\phi') - y' = \epsilon\phi'(x). \tag{29.2}$$

This gives

$$\frac{d}{dx}\delta y = \delta\frac{d}{dx}y. \tag{29.3}$$

This shows that *the derivative of the variation is equal to the variation of the derivative.*

In a similar way we may be interested in the variation of a definite integral. This means that we take the definite integral evaluated for the modified integrand minus the definite integral evaluated for the original integrand:

$$\begin{aligned} \delta\int_a^b F(x)dx &= \int_a^b \overline{F(x)}dx - \int_a^b F(x)dx \\ &= \int_a^b \left[\overline{F(x)} - F(x)\right]dx = \int_a^b \delta F(x)dx. \end{aligned} \tag{29.4}$$

Hence

$$\delta\int_a^b F(x)dx = \int_a^b \delta F(x)dx. \tag{29.5}$$

This shows that *the variation of a definite integral is equal to the definite integral of the variation.*

> *Summary.* The δ-process reveals two characteristic properties:
> (*a*) Variation and differentiation are permutable processes.
> (*b*) Variation and integration are permutable processes.

10. The stationary value of a definite integral treated by the calculus of variations. We consider once more the problem of section 7, but this time treated by the direct methods of the calculus of variations. Given the definite integral (27.5), with the boundary conditions (27.6), the stationary value of this integral is to be found.

In order to solve this problem we investigate the rate of change of the given integral, caused by the variation of the function $y = f(x)$. We start out with the variation of the integrand $F(y, y', x)$ itself, caused by the variation of y (remembering that F is a *given function* of the three variables y, y', x and this functional dependence is *not altered* by the process of variation):

$$\delta F(y, y', x) = F(y + \epsilon\phi, y' + \epsilon\phi', x) - F(y, y', x)$$
$$= \epsilon\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right). \qquad (210.1)$$

The higher order terms of the Taylor development can be neglected since ϵ approaches zero.

We are now in a position to evaluate the variation of the definite integral (27.5):

$$\delta\int_a^b F dx = \int_a^b \delta F dx = \epsilon\int_a^b\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)dx. \qquad (210.2)$$

In order to get the "rate of change" of our integral, we have to divide by the infinitesimal parameter ϵ, as we did in section 2 when the stationary value of an ordinary function $F(u_1, \ldots, u_n)$ was involved. Hence the quantity which has to vanish for a stationary value of the integral I is

$$\frac{\delta I}{\epsilon} = \int_a^b\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)dx. \qquad (210.3)$$

This expression is not accessible to further analysis in its present form, because $\phi(x)$ and $\phi'(x)$ are not independent of each other, although their relation cannot be formulated in any algebraic form. The difficulty can be removed by an ingenious application of the method of integration by parts. We can transform the second part of (210.3) as follows:

$$\int_a^b \frac{\partial F}{\partial y'}\phi' dx = \left[\frac{\partial F}{\partial y'}\phi\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\cdot\phi dx. \qquad (210.4)$$

The first term drops out since we vary between definite limits so that $\phi(x)$ vanishes at the two end-points a and b. Making use of this transformation, equation (210.3) becomes

$$\frac{\delta I}{\epsilon} = \int_a^b\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\phi dx. \qquad (210.5)$$

For the sake of brevity we introduce the following notation:

$$E(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}, \qquad (210.6)$$

and write the condition for a stationary value of I in the form

$$\int_a^b E(x)\phi(x)dx = 0. \qquad (210.7)$$

Now it is not difficult to see that this integral can vanish for *arbitrary* functions $\phi(x)$ only if $E(x)$ vanishes everywhere between a and b. Indeed, we may arrange that the function $\phi(x)$ shall vanish everywhere with the exception of an arbitrarily small interval around the point $x = \xi$. But within this interval $E(x)$ is practically constant and can be put in front of the integral sign[1]

$$\frac{\delta I}{\epsilon} = E(\xi)\int_{\xi-\rho}^{\xi+\rho}\phi(x)dx. \qquad (210.8)$$

The error we have made tends to zero as ρ tends to zero. Since the second integral is at our disposal and need not vanish, the vanishing of $\delta I/\epsilon$ requires the vanishing of the first factor. The point $x = \xi$ may be chosen as *any* point of the interval between a and b. Hence we obtain for the entire interval between a and b the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0. \qquad (210.9)$$

This condition is *necessary* for the vanishing of δI. On the other hand, it is also *sufficient*, because, if the integrand of (210.5) vanishes the integral also vanishes. *The differential equation (210.9) is thus the necessary and sufficient condition that the definite integral I shall be stationary under the given boundary conditions (27.6).*

Problem. Consider the definite integral

$$I = \int_a^b F(y, y', y'', x)\, dx, \qquad (210.10)$$

which contains the first *and second* derivatives of y. Find the condition for a stationary value by forming the variation of the integral and applying the method of integration by parts *twice*. Derive the differential equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''} = 0, \qquad (210.11)$$

[1] We assume that $E(x)$ is a *continuous* function of x.

and the boundary term

$$\delta I = \left[\left(\frac{\partial F}{\partial y'} - \frac{d}{dx}\frac{\partial F}{\partial y''}\right)\delta y + \frac{\partial F}{\partial y''}\delta y'\right]_a^b, \qquad (210.12)$$

which vanishes if y and y' are prescribed at the two end-points.

Summary. The necessary and sufficient condition for the integral

$$\int_a^b F(y, y', x)dx$$

to be stationary, with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta,$$

is that the differential equation of Euler-Lagrange

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0.$$

shall be satisfied.

11. The Euler-Lagrange differential equations for n degrees of freedom. In mechanics the problem of variation presents itself in the following form. Find the stationary value of a definite integral

$$I = \int_{t_1}^{t_2} L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)dt, \qquad (211.1)$$

with the boundary conditions that the q_k are given (and thus their variation is zero) at the two end-points t_1 and t_2:

$$[\,\delta q_k(t)\,]_{t=t_1} = 0, \quad [\,\delta q_k(t)\,]_{t=t_2} = 0. \qquad (211.2)$$

The $q_1, \ldots, q_n$ are unknown functions of t, to be determined by the condition that the actual motion shall make the integral I stationary:

$$\delta I = 0, \qquad (211.3)$$

for arbitrary independent variations of the q_k, subject only to the boundary conditions (211.2).

Now we can obviously select one definite q_k and vary it all by itself, leaving the other q_i unchanged. Hence we can apply the

differential equation (210.9) to our present problem, after adapting it to the present notation. The y corresponds to q_k, the y' to $\dot{q}_k$, the independent variable x is now the time t. The function F is denoted by L and the limits of integration are t_1 and t_2. Hence we get

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \ (t_1 \leqq t \leqq t_2). \qquad (211.4)$$

These equations have to hold for each separate q_k, k running from 1 to n.

The variations we have employed so far are but *special* variations, and the question arises whether a simultaneous variation of *all* the q_k would not bring in additional conditions. This is actually not the case, on account of the superposition principle of infinitesimal processes. Let us denote by $\delta_k I$ the variation of I produced by varying q_k alone. Then the simultaneous variation of all the q_k produces the following resultant variation:

$$\delta I = \delta_1 I + \delta_2 I + \ldots + \delta_n I. \qquad (211.5)$$

Now the differential equation (211.4) guarantees the vanishing of $\delta_k I$. If that differential equation holds for *all* indices k the sum (211.5) vanishes, and thus δI is zero for *arbitrary* variations of the q_k.

The problem of finding the stationary value of I for arbitrary variations of the q_k between definite limits is thus solved. The conditions for the stationary value of I come out in the form of the following *system of simultaneous differential equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, \ (k = 1, 2, \ldots, n). \qquad (211.6)$$

They are called "the differential equations of Euler and Lagrange," or also, if applied to problems of mechanics, "the Lagrangian equations of motion."

With the exception of the singular case in which the function L depends on some or all the $\dot{q}_k$ in a *linear* way, the partial derivatives $\frac{\partial L}{\partial \dot{q}_k}$ will contain all the $\dot{q}_k$, so that differentiation with respect to t brings in all the second derivatives $\ddot{q}_k$. We can solve the equations for the $\ddot{q}_k$ algebraically and thus rewrite the differential equations (211.6) in the following explicit form:

$$\ddot{q}_k = \phi_k(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t). \qquad (211.7)$$

The integration of such a system of differential equations of the second order involves $2n$ constants of integration, so that the complete solution of the equations (211.7) may be written as follows:

$$q_k = q_k(A_1, \ldots, A_n; B_1, \ldots, B_n; t). \qquad (211.8)$$

The constants of integration A_k and B_k can be adjusted to the given boundary conditions. Our variational problem requires variation between *definite limits*, which means that the coordinates q_k are given at $t = t_1$ and $t = t_2$. These are $2n$ boundary conditions which can be satisfied by the proper choice of the constants A_k and B_k. The nature of mechanical problems is such that more frequently *initial conditions* take the place of boundary conditions. The freedom of $2n$ constants of integration allows all the initial position coordinates and velocities to be prescribed arbitrarily.

Summary. If a definite integral is given which contains not one but n unknown functions, to be determined by the condition that the integral be stationary, we can vary these functions independently of each other. Thus the Euler-Lagrange differential equation can be formed for each function separately. This gives n simultaneous differential equations of the second order. The solution of these differential equations determines the n unknown functions in terms of the independent variable and $2n$ constants of integration.

12. **Variation with auxiliary conditions.** We consider once more the problem of the previous paragraph, with the modification that the variables $q_1, \ldots, q_n$ shall not be independent, but restricted by given auxiliary conditions. These conditions will take the form of certain functional relations between the q_k:

$$\begin{aligned} f_1(q_1, \ldots, q_n, t) &= 0, \\ &\cdots \\ &\cdots \\ &\cdots \\ f_m(q_1, \ldots, q_n, t) &= 0. \end{aligned} \qquad (212.1)$$

It is possible to eliminate m of the q_k in terms of the other variables and thus reduce the problem to $n-m$ degrees of freedom. After the reduction the differential equations of Euler-Lagrange come into play. However, this elimination may be rather cumbersome; moreover, the conditions between the variables may be of a form which makes the distinction between dependent and independent variables artificial. Here again the method of the Lagrangian multiplier, studied before in section 5, gives an adequate solution of the problem.

Variation of the equations (212.1) gives:

$$\begin{aligned} \delta f_1 &= \frac{\partial f_1}{\partial q_1}\delta q_1 + \ldots + \frac{\partial f_1}{\partial q_n}\delta q_n = 0, \\ &\cdots \\ &\cdots \\ &\cdots \\ \delta f_m &= \frac{\partial f_m}{\partial q_1}\delta q_1 + \ldots + \frac{\partial f_m}{\partial q_n}\delta q_n = 0. \end{aligned} \qquad (212.2)$$

These equations hold at any time t. According to the principle of the Lagrangian multiplier we multiply each one of these equations by an undetermined factor λ_i. Since the auxiliary conditions are prescribed for every value of the independent variable t, the λ-factors have also to be applied for every value of t, which makes them functions of t. Moreover, the summation over all the auxiliary conditions—after multiplication by λ_i—amounts to an integration with respect to the time t. Thus the method of the Lagrangian multiplier appears here in the following form: instead of putting the variation of the given integral I equal to zero, modify it in the following fashion:

$$\delta I' = \delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} (\lambda_1 \delta f_1 + \ldots + \lambda_m \delta f_m) dt = 0. \qquad (212.3)$$