Annali dell'Università di Ferrara 53: 51–64 (2007)
DOI: 10.1007/s11565-007-0005-1

Francesco Demontis · Sebastiano Pennisi

On a further condition in the macroscopic extended model for ultrarelativistic gases

Received: 3 March 2006 / Accepted: 6 November 2006 / Published online: 16 April 2007

Abstract An exact macroscopic extended model, with many moments, for ultrarelativistic gas has been recently proposed in literature. However, a further condition has not been imposed, even if it is evident in the case of a charged gas and when the electromagnetic field acts as an external force; in the present paper we exploit it and prove that it results in many identities and in residual conditions which allow to determine the arbitrary single variable functions present in the general theory. The result is that they are polynomials determined except for a corresponding number of constants. These are arbitrary constants, so that the macroscopic model remains still more general than the kinetic model.

Keywords Extended Thermodynamics · Fluid Models · Ultra-Relativistic Gas · Entropy Principle

Mathematics Subject Classification (2000) 74A15 · 74A20

1 Introduction

In order to describe the context to which this work applies, let us consider the Vlasov equation [3] multiplied by the rest particle mass m, i.e.,

$$p^{\mu}\frac{\partial f}{\partial x^{\mu}}+eF^{\alpha\mu}p_{\mu}\frac{\partial f}{\partial p^{\alpha}}=0, \tag{1.1}$$

F. Demontis
Dipartimento di Matematica ed Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail: fdemontis@unica.it

S. Pennisi (✉)
Dipartimento di Matematica ed Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail: spennisi@unica.it

where $F^{\alpha\mu}$ is the skew-symmetric electromagnetic tensor which can be decomposed as

$$\mathbf{F}^{\alpha\mu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & H_3 & -H_2 \\ -E_2 & -H_3 & 0 & H_1 \\ -E_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

in terms of the electric field E_i and the magnetic field H_i. Moreover, f is the distribution function, e the charge and p^μ the four momentum satisfying the relation $p^\mu p_\mu = -m^2$ (the light speed has been taken as unity). We neglect here the term due to collisions between the atoms, which can be found in refs. [3], [6] or [4], because it does not effect our following considerations. Obviously, if $F^{\alpha\mu} = 0$, the eq. (1.1) is the Boltzmann equation. If we multiply the eq. (1.1) by $p^{\alpha_1} \cdots p^{\alpha_n}$, with n going from 0 to a fixed N, and integrate with respect to $dP = \sqrt{-g}\frac{dp^1 dp^2 dp^3}{p_0}$ (the invariant element of momentum space), we obtain the fields equations (see the Appendix A for more details)

$$\partial_\alpha A^{\alpha\alpha_1\ldots\alpha_n} = n\, e\, F^{(\alpha_1}{}_\mu A^{\alpha_2\ldots\alpha_n)\mu}, \tag{1.2}$$

where the following definition has been used

$$A^{\alpha\alpha_1\cdots\alpha_n} = \int f p^\alpha p^{\alpha_1} \cdots p^{\alpha_n} dP, \tag{1.3}$$

which is the generic moment of the distribution function; obviously, we have the conservation laws of mass, momentum-energy. Now the entropy principle (see, for example, [5]) imposes that

$$\partial_\alpha h^\alpha = \sigma \geq 0$$

for every solution of the system (1.2), with h^α the entropy-(entropy flux) density and σ its production. This is equivalent to assume the existence of the symmetric Lagrange Multipliers $\Sigma_{\alpha_1\cdots\alpha_n}$ such that

$$\begin{aligned} & A^{\alpha\alpha_1\cdots\alpha_n} = \frac{\partial h'^\alpha}{\partial \Sigma_{\beta_1\cdots\beta_n}} P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}, \\ & \sum_{n=1}^{N} n\, e\, \Sigma_{\alpha_1\ldots\alpha_n} F^{\alpha_1}{}_\mu A^{\alpha_2\cdots\alpha_n\mu} \geq 0, \end{aligned} \tag{1.4}$$

where $h'^\alpha = \Sigma_{\alpha_1\ldots\alpha_n} A^{\alpha\alpha_1\ldots\alpha_n} - h^\alpha$, and $P^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ is the constant tensor, symmetric with respect to α_i and α_j, and such that $P^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n} g_{\alpha_1\alpha_2} = 0$. (Its presence is motivated by the fact that the trace of eq. (1.2) gives again eq. (1.2) but with $n-2$ instead of n). Now, if the electromagnetic field acts as an external force, the residual inequality $(1.4)_2$ is linear in $F_{\mu\alpha_1}$ so that it would be violated unless

$$\sum_{n=1}^{N} n A^{\alpha_2\ldots\alpha_n[\mu} \Sigma^{\alpha_1]}{}_{\alpha_2\ldots\alpha_n} = 0. \tag{1.5}$$

We want to see the implications of eq. (1.5) also for an ultra-relativistic gas. This is the first step in order to find the solution of eq. (1.5) in the case $m \neq 0$. The symmetry condition $\frac{\partial h'^{[\alpha}}{\partial \Sigma_{\beta_1 \cdots \beta_n}} P^{\alpha_1]\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n} = 0$ has been exploited in [1] and the result is that h'^{α} is determined in terms of an arbitrary function $F(X_0, X_1, \ldots, X_N)$ as

$$h'^{\alpha} = \int F(\Sigma, \Sigma_{\beta_1} p^{\beta_1}, \cdots, \Sigma_{\beta_1 \cdots \beta_N} p^{\beta_1} \cdots p^{\beta_N}) p^{\alpha} dP. \tag{1.6}$$

It remains to impose eq. (1.5) which has until now never been imposed, except for the particular case $N = 2$ (see ref. [2]). It is the object of the present paper and will be exploited in the next section. The result can be written in a simpler and elegant way; this will be the subject of the section 3 and the result is subsequent eq. (3.6) expressed in terms of a numerable family of functions $\overset{i}{G}$ restricted by eq. (3.1). This equation determines $\overset{i}{G}$ except for $\overset{0}{G}(X_0, X_1)$ and for a numerable family of functions of the single variable X_1. However, in spite of this, h'^{α} is determined except for its value at equilibrium and for a numerable family of *constants*; this is because F has to be integrated in (1.6) to give h'^{α}. The proof will be given in section 4 and the result is what we expected, because of the case $N = 2$ in [2].

2 A family of solutions of condition (1.5)

Let us consider the problem of finding a function $G(X_0, X_1, \ldots, X_N)$ such that

$$\frac{\partial F}{\partial X_n} = \frac{\partial G}{\partial X_{n+1}}, \qquad \text{with } n = 0, 1, \cdots, N-1 . \tag{2.1}$$

Obviously, this problem imposes conditions also on the function F (integrability conditions). Its importance can be seen through the following

Theorem 2.1 *If the conditions* (2.1) *are satisfied, also condition* (1.5) *holds as their consequence.*

Proof By defining $\hat{f} = \int G(X_0, X_1, \ldots, X_N) dP$ and by using eq. $(1.4)_1$ and (1.6), we obtain

$$A^{\alpha\alpha_1\ldots\alpha_n} = \int \frac{\partial F}{\partial X_n} p^{\alpha} p^{\alpha_1} \ldots p^{\alpha_n} dP = \int \frac{\partial G}{\partial X_{n+1}} p^{\alpha} p^{\alpha_1} \ldots p^{\alpha_n} dP = \frac{\partial \hat{f}}{\partial \Sigma_{\alpha\alpha_1\ldots\alpha_n}},$$

$$\text{for } n = 0, 1, \ldots, N-1, \quad \text{i.e.,}$$

$$A^{\alpha_1\ldots\alpha_n} = \frac{\partial \hat{f}}{\partial \Sigma_{\alpha_1\ldots\alpha_n}}, \text{ for } n = 1, 2, \ldots, N.$$

By using this, eq. (1.5) becomes

$$0 = \sum_{n=1}^{N} n \frac{\partial \hat{f}}{\partial \Sigma_{\alpha_2\cdots\alpha_n[\mu}} \Sigma^{\alpha_1]}{}_{\alpha_2\cdots\alpha_n} = \sum_{n=1}^{N} n \int \frac{\partial G}{\partial X_n} p^{\alpha_2} \cdots p^{\alpha_n} p^{[\mu} dP\, \Sigma^{\alpha_1]}_{\alpha_2\cdots\alpha_n} . \tag{2.2}$$

Now, the identity $0 = \int \frac{\partial}{\partial p^{\alpha_1}}(Gp^{\mu})dP$ can be expressed as

$$0 = \int \frac{\partial (Gp^{\mu})}{\partial p^{\alpha_1}} dP = \int \left(\sum_{n=1}^{N} \frac{\partial G}{\partial X_n} n \Sigma_{\alpha_1 \mu_2 \cdots \mu_n} p^{\mu_2} \cdots p^{\mu_n} p^{\mu} + G \delta^{\mu}_{\alpha_1} \right) dP ,$$

whose skew-symmetric part is exactly the condition (2.2)! Consequently to exploit the condition (1.5), it suffices to impose the eq. (2.1). □

The equation (2.1) defines the function G by integration and there remain the integrability conditions

$$\frac{\partial^2 F}{\partial X_h \partial X_{k-1}} = \frac{\partial^2 F}{\partial X_{h-1} \partial X_k}, \quad \text{for } h,k = 1, \ldots, N \ . \tag{2.3}$$

So let us exploit these conditions and let $\overset{M}{F}$ denote the homogeneous part, with degree M, of F in the variables $X_2, \ldots, X_N$. (This expansion of F is physically significant because $X_2, \ldots, X_N$ are zero at equilibrium). Obviously, $\overset{M}{F}$ can be expressed in the form

$$\overset{M}{F} = \sum_{i_1, \ldots, i_M}^{2, \ldots, N} \frac{1}{M!} f_{i_1 i_2 \ldots i_M}(X_0, X_1) X_{i_1} \cdots X_{i_M} , \tag{2.4}$$

where $f_{i_1 i_2 \ldots i_M}$ is symmetric because it multiplies the symmetric term $X_{i_1} \cdots X_{i_M}$. After that, we can see that the following theorem holds:

Theorem 2.2 *The functions $\psi^s_M(X_0, X_1)$, with s going from 1 to $(N-2)M+1$, exist such that*

$$f_{i_1 i_2 \ldots i_M} = \psi_M^{i_1 + \ldots + i_M - 2M + 1} . \tag{2.5}$$

In other words, $f_{i_1 i_2 \ldots i_M}$ depends on its indexes only through their sum! The proof of this Theorem is reported in Appendix B.

After that, we see that eq. (2.3), by use of eq. (2.4), is already satisfied for $h, k = 3, \ldots, N$. Obviously, from eqs. (2.4) and (2.5) it follows $\overset{0}{F} = \psi^1_0$. It remains to impose eq. (2.3) in the cases where at least one between h and k is less than 3. The result is expressed by the following

Theorem 2.3 *The functions ψ^r_M are determined in terms of the previous order ones, through the following eqs.* (2.6) - (2.9), *except for $N-1$ functions of the* *single variable* X_1.

$$\psi^r_M = \frac{\partial}{\partial X_1}\psi^{r+1}_{M-1}, \quad \forall r : 1 \leq r \leq N(M-1)+2-2M \text{ and } M \geq 2, \tag{2.6}$$

$$\frac{\partial}{\partial X_0}\psi^r_M = \frac{\partial}{\partial X_1}\psi^{r-1}_M, \quad \forall r : N(M-1)+3-2M \leq r \leq (N-2)M+1 \text{ and } M \geq 2, \tag{2.7}$$

$$\frac{\partial}{\partial X_0}\psi^r_1 = \frac{\partial}{\partial X_1}\psi^{r-1}_1, \quad \forall r : 2 \leq r \leq N-1\,, \tag{2.8}$$

$$\frac{\partial}{\partial X_0}\psi^1_1 = \frac{\partial^2}{\partial X_1^2}\psi^1_0. \tag{2.9}$$

In other words, eq. (2.6) *determines some of the functions* ψ^r_M; *the remaining* $N-1$ *ones are determined by eqs.* (2.7) *and* (2.8), *except for* $N-1$ *functions of the single variable* X_1, *arising from integration. At last, eq.* (2.9) *determines* ψ^1_1 *in terms of* $\overset{0}{F}$ *except for a function of* X_1, *arising from integration.*

Proof Let us begin imposing eq. (2.3) in the cases where only one between h and k is less than 3; for the skew-symmetry of eq. (2.3), it will suffice to consider the cases with $h \geq 3$. Let us begin by considering eq. (2.3) with $k=1$, $h=3,\ldots,N$ and at the order $M-1$; obviously this is possible only for $M \geq 1$. The result, by using eq. (2.4), is $\frac{\partial f_{hi_2\ldots i_M}}{\partial X_0} = \frac{\partial f_{(h-1)i_2\ldots i_M}}{\partial X_1}$, i.e., for eq. (2.5),

$$\frac{\partial}{\partial X_0}\psi^r_M = \frac{\partial}{\partial X_1}\psi^{r-1}_M, \quad \text{for every } r \text{ such that } 2 \leq r \leq (N-2)M+1. \tag{2.10}$$

In other words, each ψ^r_M (except that with the lower value of r, that is $r=1$) can be obtained from the previous one, except for an arbitrary function of the single variable X_1 arising from integration.

Let us exploit now eq. (2.3) at the order $M-2$, for $k=2$ and $h=3,\ldots,N$. By using eq. (2.4) it becomes $\frac{\partial f_{hi_2\ldots i_{M-1}}}{\partial X_1} = f_{(h-1),2,i_2\ldots i_{M-1}}$; this, for eq. (2.5), is equivalent to the above eq. (2.6). Obviously, it holds only for $M \geq 2$. The results

(2.6) and (2.10) can be described through the following two columns.

$$\begin{array}{ccc}
\psi^1_{M-1} & & \psi^1_M \\
 & \swarrow & \\
\psi^2_{M-1} & & \psi^2_M \\
\cdots & & \cdots \\
\psi^r_{M-1} & & \psi^r_M \\
 & \swarrow & \\
\psi^{r+1}_{M-1} & & \psi^{r+1}_M \\
\cdots & & \cdots \\
\psi^{N(M-1)+2-2M}_{M-1} & & \psi^{N(M-1)+2-2M}_M \\
 & \swarrow & \\
\psi^{N(M-1)+3-2M}_{M-1} & & \psi^{N(M-1)+3-2M}_M \\
\cdots & & \cdots \\
 & & \psi^{M(N-2)+1}_M
\end{array} .$$

Eq. (2.10) shows that the derivative with respect to X_0 of each element, in the two columns, is equal to the derivative with respect to X_1 of the previous one, in the same column; this holds except for the first term for which a previous one does not exist. Eq. (2.6) shows that each term of the second column is equal to the derivative with respect to X_1 of that in the first column, but in the subsequent row; this except for the terms which have not a subsequent one in the first column. The problem arises to prove that, for such terms, eq. (2.10) is an identity. This is the case, in fact, for $r = 1, \ldots, N(M-1) + 2 - 2M$ the derivative of eq. (2.6) with respect to X_0 is

$$\frac{\partial}{\partial X_0}\psi^r_M = \frac{\partial^2}{\partial X_1 \partial X_0}\psi^{r+1}_{M-1} \overset{*}{=} \frac{\partial^2}{\partial X_1^2}\psi^r_{M-1} \overset{**}{=} \frac{\partial}{\partial X_1}\psi^{r-1}_M$$

which is just eq. (2.10)! (Here the passage denoted by (*) follows from eq. (2.10) with $M-1$ instead of M and $r+1$ instead of r. Similarly, the passage denoted by (**) follows from eq. (2.6) with $r-1$ instead of r). For the remaining terms eq. (2.10) has been reported in eqs. (2.7) and (2.8). There remains to exploit eq. (2.3) when both h and k are less than 3; for the skew-symmetry of eq. (2.3) it will suffice to consider $k = 1$ and $h = 2$. In this case eq. (2.3) at the order $M-1$ and by using eq. (2.4) is

$$\frac{\partial}{\partial X_0} f_{2i_2\cdots i_M} = \frac{\partial^2}{\partial X_1^2} f_{i_2\cdots i_M},$$

which, by eq. (2.5) becomes

$$\frac{\partial}{\partial X_0}\psi^r_M = \frac{\partial^2}{\partial X_1^2}\psi^r_{M-1}, \quad \text{for every } r \text{ such that } 1 \leq r \leq (M-1)N + 3 - 2M. \tag{2.11}$$

In other words, the derivative with respect to X_0 of each element in the second column is equal to the second derivative with respect to X_1 of the corresponding element in the first column, as long as this corresponding one exist. But, for $r = 2, \dots, (M-1)N+3-2M$, from eq. (2.10) we have

$$\frac{\partial}{\partial X_0}\psi_M^r = \frac{\partial}{\partial X_1}\psi_M^{r-1} = \frac{\partial^2}{\partial X_1^2}\psi_{M-1}^r$$

(in the last passage, eq. (2.6) with $r-1$ instead of r has been used). This proves that eq. (2.10) is an identity for such values of r. It suffices then to impose it only for $r = 1$, i.e.,

$$\frac{\partial}{\partial X_0}\psi_M^1 = \frac{\partial^2}{\partial X_1^2}\psi_{M-1}^1 .$$

But, on the other hand, at least for $M > 1$, we have

$$\frac{\partial}{\partial X_0}\psi_M^1 = \frac{\partial^2}{\partial X_0 \partial X_1}\psi_{M-1}^2 = \frac{\partial^2}{\partial X_1^2}\psi_{M-1}^1$$

(in the first passage eq. (2.6) has been used, while in the second one eq. (2.10) is useful). Consequently, eq. (2.11) is always an identity, except for the case $M = 1$ and $r = 1$, which is reported in eq. (2.9). This completes the proof. □

Summarizing the results, we have that every order M is determined by $(N-2)M+1$ functions of X_0 and X_1; some of these are determined in terms of those in the previous order and the remainders $N-1$ are linked to the previous ones by (2.7) and (2.8) which determines them except for $N-1$ functions of the single variable X_1, arising from integration. In the next section these results will be expressed in a simpler and elegant way.

3 A simpler expression of the results

A simpler and more elegant way to express the previous results is to consider the numerable family of scalar functions $\overset{i}{G}(X_0, X_1)$ for i going from 0 to ∞, linked between themselves only by

$$\frac{\partial}{\partial X_0}\overset{i+1}{G} = \frac{\partial}{\partial X_1}\overset{i}{G} . \tag{3.1}$$

This determines $\overset{i+1}{G}$ except for an arbitrary function of the single variable X_1, arising from the integration with respect to X_0. The utility of these functions can be seen in the following

Theorem 3.1 *The conditions (2.6) - (2.9) in Theorem 2.3 are satisfied by the following functions*

$$\psi_M^r = \frac{\partial^M}{\partial X_1^M}\overset{r+M-1}{G} . \tag{3.2}$$

Moreover, expression (3.2) *is unique, i.e., this formula can be used without losing generality.*

Proof It is easy to see that the functions (3.2) satisfy the conditions (2.6) - (2.9), so that it remains to prove uniqueness. To this end, we proceed following an iterative procedure.

1. Firstly, we note that our property holds when $M = 0$. In fact, in the previous section no restriction has been found on $\overset{0}{F}$, the expression of F at the order zero; we have only defined $\psi_0^1 = \overset{0}{F}$. Therefore, eq. (3.2) holds with $M = 0$ (from which $r = 1$), $\overset{0}{G} = \overset{0}{F}$.
2. Note that the property holds also when $M = 1$. In fact, for this case we have found, in the preceding section, that

$$\frac{\partial}{\partial X_0}\psi_1^1 = \frac{\partial^2}{\partial X_1^2}\overset{0}{F},$$
$$\frac{\partial}{\partial X_0}\psi_1^{r+1} = \frac{\partial}{\partial X_1}\psi_1^r, \quad \text{for } r = 1, \ldots, N-2. \tag{3.3}$$

The first one of these is the integrability condition of the problem

$$\begin{cases} \dfrac{\partial \overset{1}{G}}{\partial X_0} = \dfrac{\partial \overset{0}{F}}{\partial X_1}, \text{ i.e., eq. (3.1) with } i = 0, \\ \dfrac{\partial \overset{1}{G}}{\partial X_1} = \psi_1^1, \text{ i.e., eq. (3.2) with } r = M = 1, \end{cases}$$

and guarantees the existence of the function $\overset{1}{G}$. Similarly, eq. (3.1) with $i = \tilde{r}$ and eq. (3.2) with $M = 1$, $r = \tilde{r}+1$ are

$$\begin{cases} \dfrac{\partial \overset{\tilde{r}+1}{G}}{\partial X_0} = \dfrac{\partial \overset{\tilde{r}}{G}}{\partial X_1}, \\ \dfrac{\partial \overset{\tilde{r}+1}{G}}{\partial X_1} = \psi_1^{\tilde{r}+1} \end{cases}$$

whose integrability condition is $\frac{\partial^2 \overset{\tilde{r}}{G}}{\partial X_1^2} = \frac{\partial \psi_1^{\tilde{r}+1}}{\partial X_0}$ or, because of eq. $(3.3)_2$, $\frac{\partial^2 \overset{\tilde{r}}{G}}{\partial X_1^2} = \frac{\partial \psi_1^{\tilde{r}}}{\partial X_1}$, which is the derivative with respect to X_1 of eq. (3.2) with $\tilde{r}$ instead of r and $M = 1$. In other words, if we have proved eq. (3.2) with $M = 1$ and $r \le \tilde{r}$ (and this we have done for $\tilde{r} = 1$) the above passages show that eq. (3.2) will hold also with $M = 1$ and $r = \tilde{r}+1$. This completes the proof of the case $M = 1$.
3. Let us suppose now that our property holds for $M \le \tilde{M}$ (we have already proved this when $\tilde{M} = 1$) and let us prove it for $M = \tilde{M}+1$. For $r = 1, \ldots, (N-2)\tilde{M}$ we have that

$$\psi_{\tilde{M}+1}^r = \frac{\partial}{\partial X_1}\psi_{\tilde{M}}^{r+1} = \frac{\partial^{\tilde{M}+1}}{\partial X_1^{\tilde{M}+1}} G^{r+\tilde{M}},$$

where, in the first passage eq. (2.6) of the previous section has been used, while the second passage takes into account eq. (3.2) with $\tilde{M}$ instead of M ad $r+1$ instead of r. The result proves eq. (3.2) with $\tilde{M}+1$ instead of M, but only for the above mentioned values of r. For the other values, i.e., $r = (N-2)\tilde{M}+1, \ldots, (N-2)(\tilde{M}+1)+1$, the functions $\psi^r_{\cdot}$ were restricted only by

$$\frac{\partial}{\partial X_0}\psi^r_{\tilde{M}+1} = \frac{\partial}{\partial X_1}\psi^{r-1}_{\tilde{M}+1}. \tag{3.4}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial X_0}\left(\psi^r_{\tilde{M}+1} - \frac{\partial^{\tilde{M}+1}}{\partial X_1^{\tilde{M}+1}}G^{r+\tilde{M}}\right) &= \frac{\partial}{\partial X_1}\psi^{r-1}_{\tilde{M}+1} - \frac{\partial^{\tilde{M}+2}}{\partial X_1^{\tilde{M}+1}\partial X_0}G^{r+\tilde{M}}\\ &= \frac{\partial}{\partial X_1}\left(\psi^{r-1}_{\tilde{M}+1} - \frac{\partial^{\tilde{M}+1}}{\partial X_1^{\tilde{M}+1}}G^{r+\tilde{M}-1}\right) = 0,\end{aligned}$$

where, in the first passage eq. (3.4) has been used, in the second one eq. (3.1), and the result is zero if we suppose that eq. (3.2) holds with $\tilde{M}+1$ instead of M and $r-1$ instead of r. Consequently, we have

$$\psi^r_{\tilde{M}+1} - \frac{\partial^{\tilde{M}+1}}{\partial X_1^{\tilde{M}+1}}G^{r+\tilde{M}} = \eta(X_1). \tag{3.5}$$

But $G^{r+\tilde{M}}$ was obtained from eq. (3.1), i.e., from $\frac{\partial}{\partial X_0}G^{r+\tilde{M}} = \frac{\partial}{\partial X_1}G^{r+\tilde{M}-1}$, except for an arbitrary function of the single variable X_1, i.e.,
$G^{r+\tilde{M}} = \hat{G}^{r+\tilde{M}}(X_0, X_1) + \mu(X_1)$.
Therefore, if we choose $\mu(X_1)$ between the solutions of $\frac{d^{\tilde{M}+1}}{dX_1^{\tilde{M}+1}}\mu(X_1) = -\eta(X_1)$, it follows $\psi^r_{\tilde{M}+1} = \frac{\partial^{\tilde{M}+1}}{\partial X_1^{\tilde{M}+1}}\hat{G}^{r+\tilde{M}}$, which is eq. (3.2) with $\tilde{M}+1$ instead of M and $\hat{G}^{r+\tilde{M}}$ instead of $G^{r+\tilde{M}}$. Obviously, the hat (^) can be omitted because it was sufficient, for our purposes, to find one of these functions. In this way, by supposing that eq. (3.2) holds with $\tilde{M}+1$ instead of M and $r-1$ instead of r, we have proved that it holds also for the same index r.

This completes our proof of uniqueness. □

Before ending this section, we note from eq. (3.2) that to determine F up to the order M, we need only the $(N-1)M+1$ functions $\overset{0}{G}, \overset{1}{G}, \ldots, \overset{(N-1)M}{G}$. In other words, even if the functions $\overset{i}{G}$ are elements of a numerable family of functions, up to the order M only a finite number of them occur, i.e., $(N-1)M+1$. From eqs. (2.4), (2.5) and (3.2) it follows

$$F = \sum_{M=0}^{\infty}\sum_{i_1,\ldots,i_M}^{2,\ldots,N}\frac{1}{M!}\left(\frac{\partial^M}{\partial X_1^M}G^{i_1+\cdots+i_M-M}\right)X_{i_1}\cdots X_{i_M}. \tag{3.6}$$

A further simplification will occur for h'^{α} and will be shown in the next section.

4 The contribute to h'^{α} of the arbitrary functions of X_1 arising by integrating eq. (3.1)

Let us see the effects of the arbitrary functions, of the single variable X_1, arising from integration of eq. (3.1). They are expressed by the following two theorems:

Theorem 4.1 *The function h'^{α}, is determined except for a numerable family of constants, even if the functions $\overset{i}{G}$ in eq. (3.1) are determined except for a numerable family of single variable functions.*

Theorem 4.2 *A particular solution of eq. (3.1) is given by*

$$\overset{0}{G} = G(X_0 + X_1), \tag{4.1}$$
$$\overset{i}{G} = G(X_0 + X_1) + \sum_{k=1}^{i} c_k \left(\frac{d^{i-k}}{dX_1^{i-k}} \frac{e^{-X_1}}{X_1^k} \right) \frac{X_0^{i-k}}{(i-k)!},$$

where $G(X)$ is an arbitrary single variable function and $\{c_k\}$ is a numerable family of arbitrary constants.

Proof (Of Theorem 4.1.) Let us call $\overset{\bar{i}}{G}_s(X_1)$ the arbitrary function arising from integration of eq. (3.1), for $i = \bar{i} - 1$, with a given $\bar{i}$; it contributes to $\overset{i}{G}$ for $i \geq \bar{i}$ the supplementary term

$$G_s^i = \frac{X_0^{i-\bar{i}}}{(i-\bar{i})!} \frac{\partial^{i-\bar{i}}}{\partial X_1^{i-\bar{i}}} G_s^{\bar{i}}(X_1),$$

how it can be easily seen with the iterative procedure. Consequently, it contributes to F, according to eq. (3.6), the supplementary term

$$F_s = \sum_{M=0}^{\infty} \sum_{i_1,\ldots,i_M}^{2,\ldots,N} \frac{1}{M!} \left[\frac{\partial^{i_1+\ldots+i_M-\bar{i}}}{\partial X_1^{i_1+\ldots+i_M-\bar{i}}} G_s^{\bar{i}}(X_1) \right] \frac{X_0^{i_1+\ldots+i_M-M-\bar{i}}}{(i_1+\ldots+i_M-M-\bar{i})!} X_{i_1} \cdots X_{i_M}, \tag{4.2}$$

where, only the terms with $i_1 + \ldots + i_M \geq \bar{i} + M$ have to be considered. Let us see how it contributes to h'^{α} according to eq. (1.6). But, firstly, we have to evaluate this equation in one of the reference frames where $\Sigma_\beta \equiv (\gamma, 0, 0, 0)$ with $\gamma = \sqrt{-\Sigma_\beta \Sigma^\beta}$. Let us also define $p^0 = \varrho$, $q^\beta = \frac{1}{\varrho} p^\beta$, from which we have $q^0 = 1$ and, moreover, $p^\beta p_\beta = 0$ becomes $q^i q_i = 1$. We can introduce spherical coordinates, i.e., $p^1 = \varrho \sin\theta \cos\phi$, $p^2 = \varrho \sin\theta \sin\phi$, $p^3 = \varrho \cos\theta$ (from which $q^1 = \sin\theta\cos\phi$, $q^2 = \sin\theta \sin\phi$, $q^3 = \cos\theta$) with $0 \leq \varrho$, $0 \leq \theta < \pi$, $0 \leq \phi \leq 2\pi$. In this way, the integral (1.6) becomes

$$h'^{\alpha} = \int_0^\infty d\varrho \int_0^{2\pi} d\phi \int_0^\pi F\left(\Sigma, \gamma\varrho, \ldots, \Sigma_{\beta_1 \cdots \beta_N} q^{\beta_1} \cdots q^{\beta_N} \varrho^N\right) q^\alpha \sqrt{-g} \varrho^2 \sin\theta \, d\theta.$$

Now we can see that the term (4.2) contributes to h'^{α} the supplementary term

$$h'^{\alpha}_s = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \sum_{M=0}^{\infty} \sum_{i_1,\ldots,i_M}^{2,\ldots,N} \frac{1}{M!} \left[\frac{\partial^{i_1+\ldots+i_M-\bar{i}}}{\partial X_1^{i_1+\ldots+i_M-\bar{i}}} G_s^{\bar{i}}(X_1) \right]_{X_1=\gamma\varrho} \varrho^{i_1+\cdots+i_M+2}$$
$$\cdot \frac{X_0^{i_1+\ldots+i_M-M-\bar{i}}}{(i_1+\ldots+i_M-M-\bar{i})!} Y_{i_1}\cdots Y_{i_M} q^{\alpha}\sqrt{-g}\sin\theta\, d\varrho,$$

where $Y_i = \Sigma_{\beta_1\cdots\beta_i} q^{\beta_1}\cdots q^{\beta_i} = \varrho^{-i} X_i$. This term can also be written as

$$h'^{\alpha}_s = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sum_{M=0}^{\infty} \sum_{i_1,\ldots,i_M}^{2,\ldots,N} \frac{1}{M!} \frac{X_0^{i_1+\ldots+i_M-M-\bar{i}}}{(i_1+\ldots+i_M-M-\bar{i})!} Y_{i_1}\cdots Y_{i_M}$$
$$\cdot q^{\alpha}\sqrt{-g}\sin\theta\, k^{i_1+\cdots+i_M}\gamma^{-(i_1+\cdots+i_M+3)},$$

with

$$k^{i_1+\cdots+i_M} = \int_0^{\infty} \left[\frac{\partial^{i_1+\ldots+i_M-\bar{i}}}{\partial X_1^{i_1+\ldots+i_M-\bar{i}}} G_s^{\bar{i}}(X_1) \right]_{X_1=\gamma\varrho} (\gamma\varrho)^{i_1+\cdots+i_M+2}\gamma\, d\varrho.$$

By using the change of variable $\varrho = \frac{X_1}{\gamma}$, this last one becomes

$$k^{i_1+\cdots+i_M} = \int_0^{\infty} \left[\frac{\partial^{i_1+\ldots+i_M-\bar{i}}}{\partial X_1^{i_1+\ldots+i_M-\bar{i}}} G_s^{\bar{i}}(X_1) \right] X_1^{i_1+\cdots+i_M+2}\, dX_1, \tag{4.3}$$

which is a constant. Obviously, $G_s^{\bar{i}}(X_1)$ must be such that the integral be convergent. The eq. (4.3) can be written as

$$k^i = \int_0^{\infty} \left[\frac{\partial^{i-\bar{i}}}{\partial X_1^{i_1-\bar{i}}} G_s^{\bar{i}}(X_1) \right] X_1^{i+2}\, dX_1, \quad \text{for } i \geq \bar{i}+M.$$

But, integrating by parts, we have

$$k^i = \left| \left[\frac{\partial^{i-\bar{i}-1}}{\partial X_1^{i-\bar{i}-1}} G_s^{\bar{i}}(X_1) \right] X_1^{i+2} \right|_0^{\infty} - \int_0^{\infty} \left[\frac{\partial^{i-\bar{i}-1}}{\partial X_1^{i-\bar{i}-1}} G_s^{\bar{i}}(X_1) \right] X_1^{i+1}(i+2)\, dX_1 =$$
$$-(i+2)k^{i-1},$$

from which

$$k^i = (-1)^r (i+2)(i+1)\ldots(i+3-r)k^{i-r} = (-1)^r \frac{(i+2)!}{(i+2-r)!} k^{i-r}.$$

This last one, for $r = i - \bar{i} - M$ becomes

$$k^i = (-1)^{i-\bar{i}-M} \frac{(i+2)!}{(\bar{i}+2+M)!} k^{\bar{i}+M}.$$

This result shows that all these constants are determined in terms of that with a lower order number of indexes, i.e.,

$$k^{\bar{i}+M} = \int_0^\infty \left[\frac{\partial^M}{\partial X_1^M} \overset{\bar{i}}{G_s}(X_1) \right] X_1^{\bar{i}+M+2} dX_1 .$$

But we can continue to integrate by parts, obtaining

$$k^{\bar{i}+M} = (-1)^M \frac{(\bar{i}+2+M)!}{(\bar{i}+2)!} \int_0^\infty \overset{\bar{i}}{G_s}(X_1) X_1^{\bar{i}+2} dX_1, \tag{4.4}$$

which is a constant; it is also arbitrary because if $k^{\bar{i}+M}$ is assigned, we can choose

$$\overset{\bar{i}}{G_s} = \frac{1}{2}(-1)^M \frac{(\bar{i}+2)!}{(\bar{i}+2+M)!} k^{\bar{i}+M} e^{-X_1} X_1^{-\bar{i}}, \tag{4.5}$$

and eq. (4.4) will become an identity. □

We note also that the expression (4.5) guarantees also the convergence of all the previous integrals and the calculations performed regarding them.

So we have found that every arbitrary function of the single variable X_1 (arising by integrating eq. (3.1)) contributes to h'^α a term determined except for the arbitrary constant $k^{\bar{i}+M}$.

Proof (Of Theorem 4.2.) By substituting eq. (4.1) in eq. (3.1), it is easy to see that these last ones are satisfied. □

5 Conclusions

We consider very interesting all these results, firstly because they fill a gap in a literature on this subject. Only particular cases ($N = 2$) have been studied previously and for this case the present results coincides with those already known. Here we have considered the macroscopic model and it includes the kinetic one [1] as the particular case with $F(X_0, X_1, \ldots, X_N) = \tilde{F}(X_0 + X_1 + \ldots + X_N)$, so that $f_{i_1 i_2 \ldots i_M} = \left(\frac{\partial^M F}{\partial X_{i_1} \ldots \partial X_{i_M}} \right)_{eq.} = \tilde{F}^{(M)}(X_0 + X_1)$. This result, for eq. (2.5) implies $\psi^s_M = \tilde{F}^{(M)}(X_0 + X_1)$ for all values of s. Finally, eq. (3.2) yields $\overset{i}{G} = \tilde{F}(X_0 + X_1)$ for all values of i. This fact gives a further confirmation to the present results, because it is well known from other contexts that the kinetic approach is more restrictive than the macroscopic one which we have here considered. It is interesting that also the particular solution (4.1) is still more general than the kinetic approach; this last one is the particular case with $\tilde{F} = G$ and $c_k = 0$. There is the possibility that eq. (4.1) is also the general solution, but we leave this investigation for future considerations.

A Appendix: proof of eq. (1.2)

In order to prove eq. (1.2), let us consider the Vlasov equation (1.1). Multiplying this equation by $p^{\alpha_1}\cdots p^{\alpha_n}$ and integrating with respect to $dP = \sqrt{-g}\,\frac{dp^1dp^2dp^3}{p_0}$, we find

$$\int p^{\alpha_1}\cdots p^{\alpha_n} p^\mu \frac{\partial f}{\partial x^\mu} dP + eF^{\alpha\mu}\int p_\mu p^{\alpha_1}\cdots p^{\alpha_n}\frac{\partial f}{\partial p^\alpha} dP = 0. \tag{A.1}$$

It is easy to observe that the first integral in the left-hand side of eq. (A.1) can be written as

$$\frac{\partial}{\partial x^\mu}\int f p^{\alpha_1}\cdots p^{\alpha_n} p^\mu \, dP,$$

because $p^{\alpha_1}\cdots p^{\alpha_n}$ do not depend on x^μ. Moreover, the second integral in the first member of eq. (A.1) becomes

$$\begin{aligned} &eF^\alpha{}_\mu\int \frac{\partial}{\partial p^\alpha}(p^\mu p^{\alpha_1}\cdots p^{\alpha_n} f)\, dP - eF^\alpha{}_\mu\int \frac{\partial}{\partial p^\alpha}(p^\mu p^{\alpha_1}\cdots p^{\alpha_n}) f\, dP \\ &= eF^\alpha{}_\mu\int \frac{\partial}{\partial p^\alpha}(p^\mu p^{\alpha_1}\cdots p^{\alpha_n} f)\, dP - eF^\alpha{}_\mu\int (n+1) g^{(\mu}_\alpha p^{\alpha_1}\cdots p^{\alpha_n)} f\, dP. \end{aligned} \tag{A.2}$$

Because in the kinetic theory is assumed that the distribution function f is such that

$$\int f p^\mu p^{\alpha_1}\cdots p^{\alpha_n} < \infty$$

the first integral in the second member of eq. (A.2) is equal to zero, while the second integral in the second member of eq. (A.2) can be expressed as follows

$$-eF^\alpha{}_\mu g^\mu_\alpha \int f p^{\alpha_1}\ldots p^{\alpha_n} dP - neF^\alpha{}_\mu \int f g^{(\alpha_1}_\alpha p^{\alpha_2}\cdots p^{\alpha_n)} p^\mu dP. \tag{A.3}$$

In the preceding expression the first term is equal to zero because $F^{\alpha\mu}$ is skew-symmetric and, hence, has zero trace. Then, eq. (A.1) can be written as

$$\frac{\partial}{\partial x^\mu}\int f p^{\alpha_1}\cdots p^{\alpha_n} p^\mu \, dP - neF^\alpha{}_\mu \int f g^{(\alpha_1}_\alpha p^{\alpha_2}\cdots p^{\alpha_n)} p^\mu dP = 0 \tag{A.4}$$

which is, recalling definition (1.3), equivalent to

$$\partial_\mu A^{\mu\alpha_1\cdots\alpha_n} - neF^{(\alpha_1}{}_\mu A^{\alpha_2\cdots\alpha_n)\mu} = 0.$$

This equation coincides with eq. (1.2) and this completes the proof.

B Appendix: proof of Theorem 2.2

Obviously, eq. (2.5) is trivial when $M = 0, 1$; in fact, for M=0 it amounts to $\overset{0}{F} = \psi_0^1$ which can be considered the definition of ψ_0^1, while for $M = 1$ it amounts to $f_{i_1} = \psi_1^{i_1-1}$ which can be considered the definitions of $\psi_1^1, \psi_1^2, \ldots, \psi_1^{N-1}$. Then it remains to prove eq. (2.5) when $M \geq 2$. Eq. (2.3) at the order $M-2$ and with $h, k = 3, \ldots, N$ and by using eq. (2.4) gives

$$f_{h,k-1,i_3,\ldots,i_M} = f_{h-1,k,i_3,\ldots,i_M}. \tag{B.5}$$

This means that we can subtract an unity from an index (which is not 2) and add it to another index (which is not N); the result remains the same. Then we can proceed in the following way: Let us choose a couple of indexes which are not 2, neither N; after that, let us subtract an unity from the lower one and add it to the greater one. We repeat this procedure more times until that the lower index becomes 2 or the greater one becomes N. After that, we can do the same thing with another couple of indexes. At the end, we will find that $f_{i_1,\dots,i_M}$ is equal to one of the following terms

$$f_{\underbrace{2\dots2}_{j\text{ times}}\underbrace{N\dots N}_{M-j-1\text{ times}}k}, \quad \text{with } k=3,\dots,N \text{ and } j=0,\dots,M. \tag{B.6}$$

We prove now that the sum of the indexes of each of these term is different from that of the other ones and, therefore, identifies this term (Note that the above procedure leaves unchanged the sum of the indexes). In other words, let us see if it is possible that $2j+(M-j-1)N+k=2j'+(M-j'-1)N+k'$. This is equivalent to

$$k-k'=(N-2)(j-j'), \tag{B.7}$$

from which $|k-k'|=(N-2)|j-j'|$ from which it follows

$$\begin{cases}|k-k'|\geq N-2 & \text{if } |j-j'|\geq 1,\\ |k-k'|=0 & \text{if } |j-j'|=0.\end{cases}$$

But

$$\begin{cases}k=3,\dots,N\\ k'=3,\dots,N\end{cases} \qquad \text{so that } |k-k'|\leq N-3.$$

Consequently, it is not possible that $|j-j'|\geq 1$, otherwise we would have $N-2\leq|k-k'|\leq N-3$ which is not possible; then we have $|j-j'|=0$, from which $j=j'$ and from eq. (B.7) it follows $k=k'$; this completes the proof.

At last, let us see that the sum of the indexes in the terms (B.6) is whatever number r between $2M$ and NM. In fact, from

$$2j+(M-j-1)N+k=r, \tag{B.8}$$

we have $(N-2)j+N-k=MN-r$ whose right-hand side, thanks to $2M\leq r\leq NM$, is any number between 0 and $(N-2)M$; it follows that $j=\left[\frac{MN-r}{N-2}\right]$ and $N-k$ is the remainder of the division of $MN-r$ and $N-2$ (to this end let us remember that $0\leq N-k<N-2$, because $2<k\leq N$).

Now that this other proof is completed, we can define

$$\psi_M^s=f_{\underbrace{2\dots2}_{j\text{ times}}\underbrace{N\dots N}_{M-j-1\text{ times}}k},$$

with j, k and s defined in terms of r by eq. (B.8) and by $s=r-2M+1$. From $2M\leq r\leq NM$ it follows $1\leq s\leq(N-2)M+1$. This allows to deduce eq. (2.5) from eq. (B.5) and (B.6).

References

1. Borghero, F., Demontis, F., Pennisi, S.: *An exact macroscopic extended model with many moment for ultra-relativistic gases*. Proceedings 'Wascom 2003' 12^{th} Conference on waves and stability in continuous media. Word Scientific-Singapore, 94–101 (2004)
2. Borghero, F., Pennisi, S.: The non-linear macroscopic model of Relativistic Extended Thermodynamics of an ultra-relativistic gas. Rend. Mat. Acc. Lincei **s.9, v.15**, 59-68 (2004)
3. Chapman, S.C., Cowling, T.G.: *The mathematical theory of nonuniform gases*. Cambridge University Press, London and New York (1961)
4. Müller, I.: *Thermodynamics*. Pitman, New York (1985)
5. Müller, I., Ruggeri, T.: *Rational Extended Thermodynamics*. Second Edition, Springer Verlag, New York, Berlin Heidelberg (1998)
6. Truesdell, C., Munaster, R.G.: *Fundamentals of Maxwell's kinetic theory of a simple monoatomic gas*. Academic Press, New York (1980)