Solitons and Inverse Scattering Transform: a brief introduction

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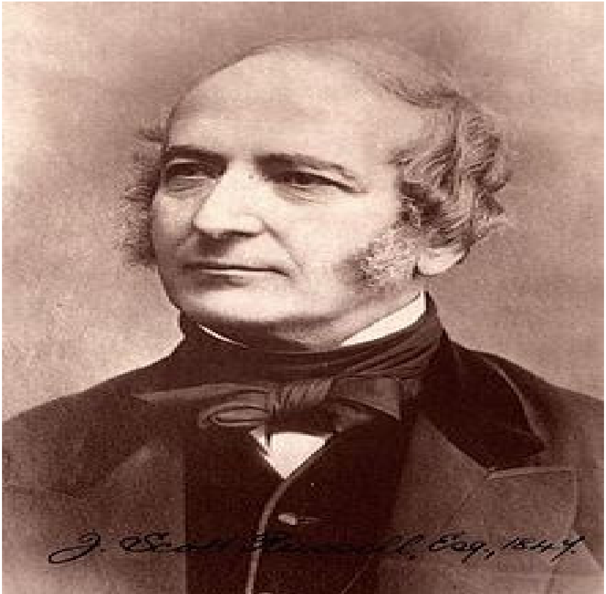
Scott Russell



Scott Russell

"...I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stop- not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still on a rate of some eight or nine miles an hour, preserving its original figure....in the month of August 1834 was my first chance inteview with that singular and beatiful phenomenon which I have called the Wave of Translation.....The fisrt day I saw it it was the happiest day of my life"

Russell's experiment

Scott Russell had success in reproducing what he saw and, moreover, he got the following (empirical) relation

$$c^2 = g(h + \eta),$$

where g represents gravity, h the channel's deep and η the maximum height of the wave.

- Airy e Stokes
- Boussinesq (1877)

Russell's experiment

Korteweg-de Vries

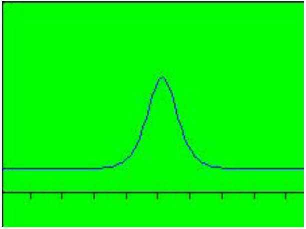
Korteweg and De Vries obtained the following equation (known as KdV equation):

$$u_t + u_{xxx} - 6uu_x = 0$$

which describes wave propagation (in one dimension) where the amplitude of the wave is small with respect to the other quantities (deep and wave length).

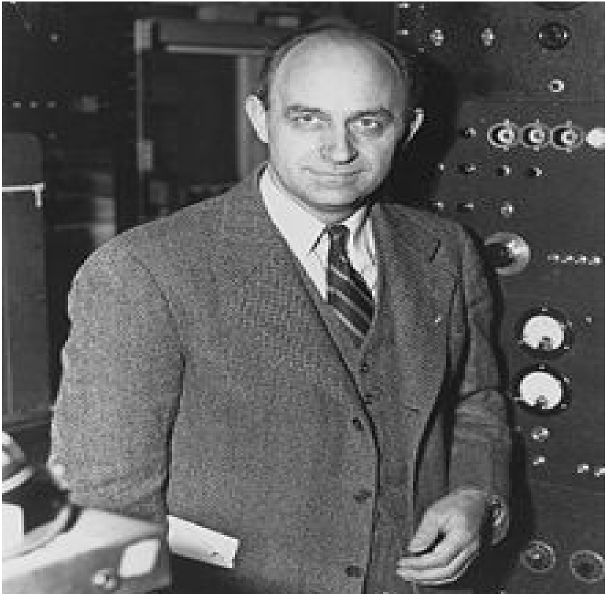
Moreover, they found a class of exact solution of this equation

$$u(x,t) = \frac{-\frac{1}{2}c}{\cosh^2 x - ct - a}.$$



....Scott Russell was right!

Fermi-Pasta-Ulam's puzzle and Mary Tsingou



Fermi-Pasta-Ulam's puzzle and Mary Tsingou

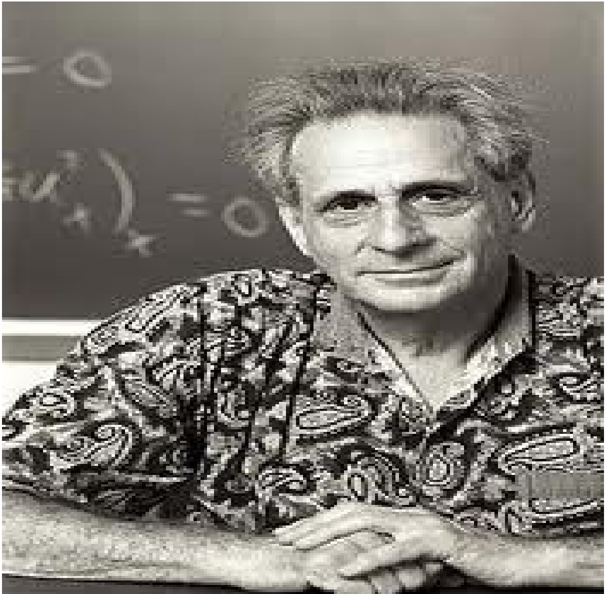
...Debye (1914)...

FPUT tried to study numerically a system with 64 (**sixty-four**) springs connected in a nonlinear way. This system is described from the following difference equation:

$$m\ddot{x}_j = k(x_{j+1} + x_{j-1} - 2x_j)[1 + \alpha(x_{j+1} - xj - 1)], \quad j = 0, 1, \ldots, 63.$$

They were sure to obtain equipartition of the energy between the springs, instead...

Kuskal and Zabusky



Kuskal and Zabusky

In 1965, Kruskal and Zabusky, solved the FPUT's puzzle.

They observed that, taking the limit in an appropriate way, the FPUT's model, is given by the KdV equation,

$$u_t + u_{xxx} - 6uu_x = 0$$

which admits a soliton solution. By using initial periodic conditions, Kruskal and Zabusy justified the numerical results obtained by FPUT!

They introduced the word soliton.
How soliton interact each other?

Solitons and...Cinema

Many thanks Barbara Prinari who gave me the following movie

1967: KdV e Inverse Scattering Transform

In 1967 Gardner, Greene, Miura e Kruskal, in order to solve the initial value problem for the KdV equation, introduced a method known as Inverse Scattering Transform (which could be considered the analogous of the Fourier transform for linear ODE).
The IST is not a direct method...it works by associating the Schroedinger equation to the Cauchy problem of the KdV:

$$-\psi_{xx} + u(x,0)\psi = \lambda^2\psi,\ x \in \mathbb{R}.$$

$$\begin{tikzcd}
\boxed{\text{given } u(x,0)} \arrow[rr, "\substack{\text{direct scattering problem}\\ \text{with potential } u(x,0)}"] \arrow[d, "\text{Solution NPDE}"] & & \boxed{S(\lambda,0)} \arrow[d, "\substack{\text{time evolution of}\\ \text{scattering data}}"'] \\
\boxed{u(x,t)} & & \boxed{S(\lambda,t)} \arrow[ll, "\substack{\text{inverse scattering problem}\\ \text{with time evolved scattering data}}"]
\end{tikzcd}$$

1967: KdV e Inverse Scattering Transform

GGKM were also lucky because three years before their work, Faddeev in 1964 had had success in solving the inverse problem for the Schroedinger equation.

1. **Direct Scattering** consists of: Find the so-called scattering data $\{R(k), \{\kappa_j, N_j\}_{j=1}^{N}\}$ of the Schroedinger equation with potential $u(x,0)$.

$$f_r(k,x) = \begin{cases} \frac{1}{T(k)} e^{-ikx} + \frac{R(k)}{T(k)} e^{ikx} + o(1), & x \to +\infty, \\ e^{-ikx}[1+o(1)], & x \to -\infty. \end{cases}$$

2. **Propagation of scattering data**: the scattering data evolve in time following the equations

$$\{R(k), \{\kappa_j, N_j\}_{j=1}^{N}\} \mapsto \{R(k)e^{8ik^3 t}, \{\kappa_j, N_j e^{8\kappa_j^3 t}\}_{j=1}^{N}\}.$$

3. **Inverse scattering** consists of: (Re)-construct the potential. To do that: 1) Solve this Marchenko equation

$$K(x,y) + \Omega(x+y) + \int_x^\infty dz\, K(x,z)\Omega(z+y) = 0,$$

where $\Omega(x) = \sum_{j=1}^{N} N_j e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} R(k)$ and
2) get $u(x,t)$ from the relation $u(x,0) = 2\frac{d}{dx}K(x,x)$.

Inverse Scattering Transform: only luck?

Few years later, it became clear that many other nonlinear evolution equation could be solved by the IST:

- Nonlinear Schroedinger equation (1972, Zakharov and Shabat)
- sine-Gordon (1973-74, Ablowitz, Kaup, Newell, Segur or Zakharov)
- Manakov system (1973, Manakov)
- AKNS system (1974, Ablowitz, Kaup, Newell, Segur)
- Camassa-Holm equation (1992, Camassa e Holm)
- Degasperis-Procesi equation (1999, Degasperis e Procesi)
- the list is not complete...

Examples of integrable equation

$u_t + 6uu_x + u_{xxx} = 0, \quad$ *Korteweg-de Vries (KdV) equation*

$u_{xt} = \sin u, \quad$ *sine-Gordon equation*

$i\, u_t + u_{xx} \pm 2uu^{\dagger} u = 0, \quad$ *Nonlinear Schrödinger (NLS) equation,*

$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad$ *Camassa-Holm (CH) equation,*

$u_{zz} + \dfrac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = 0, \quad$ *Kadomtsev-Petviashvily (KP) equation.*

IST

We call *integrable* each Nonlinear PDE to which can be applied the **I**nverse **S**cattering **T**ransform.
The IST is a very powerful method which allow one to solve the Cauchy problem of many "different" Nonlinear PDE.
All the equations (systems) solvable by IST are caracterized by important properties such as:

- They admit a class of exact solution: an interesting sub-class of this type of solution is given by the *soliton solutions* .
- They are *integrable Hamiltonian systems*, in the sense that if the IST is applicable then it is possible to find a canonical transformation from physical variables to angle-action variables.
- They have an infinite number of conserved quantitities ($\int f(x,t)dx$ is a conserved quantity if $\frac{d}{dt}\int f(x,t)dx = 0$).

Lax pair

Establishing if a given PDE is integrable (in the sense specified before) is not a trivial task. But, if we know that a system of PDE is associated to a LODE system, we can establish its integrability by using the method of the Lax pair.

Lax's method can be described as follows:

Given a system of LODE L, find A such that

$$L_t + LA - AL = 0.$$

By imposing the last equation, we obtain the NPDE associated with the system of LODE (from which we departed).

Integrability: One example

1. Given

$$L = -\frac{d^2}{dx^2} + u(x,t)$$

try to determine A. If we look for an operator A of the following type:

$$A = \alpha_3 \frac{d^3}{dx^3} + \alpha_2 \frac{d^2}{dx^2} + \alpha_1 \frac{d}{dx} + \alpha_0,$$

the compatibility (Lax pair) condition $L_t + LA - AL = 0$ leads (choosing $\alpha_3 = -4, \alpha_2 = 0, \alpha_1 = 6u, \alpha_0 = 3u_x$) to the equation $u_t - 6uu_x + u_{xxx} = 0$, i.e. the KdV equation.

Recent developments: the triplet method

Problem: Determine explicit solutions of the (m)NLS equation. We are mainly interested in an important sub-class of this (explicit) class of solutions: the soliton solutions

Several methods has been developed to determine analytic solutions of the NLS equations. Solutions obtained using one of this method not necessarily can be reconstructed by following another of these methods.

Development of a method, based on the Inverse Scattering Transform (IST), which is able to give- in a unified way- the solutions obtained by using the methods so far used.

IST for the (m)NLS equation

1. Consider the initial value problem for the (m)NLS equation:

$$\begin{cases} i\, u_t + u_{xx} + 2uu^{\dagger}u = 0, \\ u(x,0). \end{cases}$$

2. The following system of ODE (AKNS system) is related to the (m)NLS equation:

$$-iJ\frac{d}{dx}\Psi(x,\lambda) - i\underbrace{\begin{pmatrix} 0_{n\times n} & q(x) \\ q(x)^{\dagger} & 0_{m\times m} \end{pmatrix}}_{\text{potential } V(x)}\Psi(x,\lambda) = \lambda\Psi(x,\lambda),$$

where λ is the spectral parameter, $x \in \mathbb{R}$, and $J = I_n \oplus (-I_m)$. Moreover, $q(x)$ is a matrix funcion with elements in $L^1(\mathbb{R})$.

$$\begin{tikzcd}
\boxed{\text{given } u(x,0)} \arrow[rr, "\substack{\text{direct scattering problem} \\ \text{with potential } u(x{,}0)}"] \arrow[d, "\text{Solution NPDE}"] & & \boxed{S(\lambda,0)} \arrow[d, "\substack{\text{time evolution of} \\ \text{scattering data}}"'] \\
\boxed{u(x,t)} & & \boxed{S(\lambda,t)} \arrow[ll, "\text{inverse scattering problem}"]
\end{tikzcd}$$

Explicit Solutions

This method is based on the IST and allows us to write the solution through a matrix triplet (A, B, C) (of dimensions, respectively, $p \times p$, $p \times m$ e $n \times p$) and the matrix exponential. This solution has the following form:

$$u(x,t) = -2CG(x,t)^{-1}B,$$

where

$$\begin{aligned}
G(x,t) &= e^{-\beta} + N\, e^{\beta^\dagger} Q, \\
\beta &= 2Ax + 4iA^2 t, \\
Q &= \int_0^\infty d\gamma\, e^{-\gamma A^\dagger} C^\dagger C e^{-\gamma A}, \\
N &= \int_0^\infty d\beta\, e^{-\beta A} B B^\dagger e^{-\beta A^\dagger},
\end{aligned}$$

Examples

It is easy to plot the soliton solutions obtained.

For example, choosing $A = (p + iq)$ where $p > 0$, $B = (1)$, $C = (c)$ with $c \neq 0$. Computing

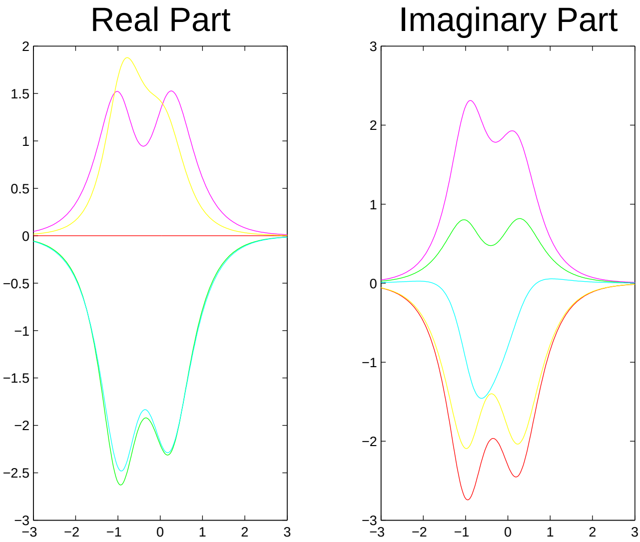
$$Q = \int_0^\infty dy\, e^{-(p-iq)y} |c|^2 e^{-(p+iq)y} = \frac{|c|^2}{2p}, \qquad N = \frac{1}{2p}.$$

Then

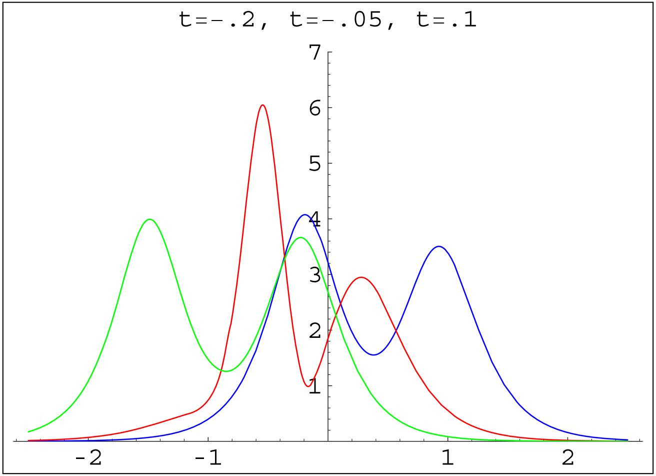
$$\begin{aligned} q(x,t) &= \frac{-2c\, e^{4i[p^2-q^2+2pqi]t} e^{-2x[p+iq]}}{1 + \frac{|c|^2}{4p^2}\, e^{-4px} e^{-4i[p^2-q^2-2pqi]t} e^{4i[p^2-q^2+2pqi]t}} \\ &= -\frac{c}{|c|} e^{2iqx_0} \frac{2p\, e^{i[2q(x-x_0+4qt)+4(p^2+q^2)t]}}{\cosh[2p(x-x_0+4qt)]}, \end{aligned}$$

where $2px_0 = \ln(|c|/2p)$. This the well-known bright soliton solution.

Two-soliton solution for $A = \mathrm{diag}(1, 2)$, $B = (1, 1)^T$, and $C = (3, 2)$.



Two-soliton solution for $A = \begin{pmatrix} 2-i & -1 \\ 0 & 2-i \end{pmatrix}$, $B = (0,1)^T$, and $C = (1+2i, -1+4i)$. Absolute value



Advantages

The advantages to have a solution formula such as this introduced before are several, for example:

- Possibility to get many of the solutions obtained in literature by using different methods as special case of a unique formula.
- Generation of a new solutions.
- Solution test useful to verify algorithimics used to solve numerically the NLS equation.
- Extension of the solution formula to matrix NLS system. It is important to remark that (m)NLS is important because it describes physical system where the electric field has two components transversal to the direction of propagation (in optical fiber).

The triplet method

Let us consider the AKNS system

$$-iJ\frac{d}{dx}\Psi(x,\lambda) - i\underbrace{\begin{pmatrix} 0_{n\times n} & q(x) \\ q(x)^\dagger & 0_{m\times m} \end{pmatrix}}_{\text{potential } V(x)}\Psi(x,\lambda) = \lambda\Psi(x,\lambda),$$

where $J = I_n \oplus (-I_m)$, $q(x) \in L^1(\mathbb{R})$.

The IST can be described as follows:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{given}\\ u(x,0)=q(x)\end{array}} \arrow[r, "\substack{\text{direct scattering problem}\\ \text{with potential } u(x,0)}"] \arrow[d, "\text{IST}"] & \boxed{\begin{array}{c}R(\lambda), \lambda_j, \Gamma_{lj}\\ \text{for } j=1,\ldots,N\end{array}} \arrow[d, "\substack{\text{time evolution of}\\ \text{scattering data}}"'] \\
\boxed{u(x,t)} & \boxed{e^{4i\lambda^2 t}R(\lambda), \lambda_j, e^{-4i\lambda_j^2 t}\Gamma_{lj}} \arrow[l, "\substack{\text{inverse scattering problem}\\ \text{with time evolved scattering data}}"]
\end{tikzcd}$$

Direct Scattering Problem

$$F_l(x,\lambda) \simeq \begin{cases} e^{i\lambda xJ}, & x \to +\infty, \\ e^{i\lambda xJ} a_l(\lambda), & x \to -\infty, \end{cases}$$

$$F_r(x,\lambda) \simeq \begin{cases} e^{i\lambda xJ}, & x \to -\infty, \\ e^{i\lambda xJ} a_r(\lambda), & x \to +\infty. \end{cases}$$

$$F_l(x,\lambda) = e^{i\lambda Jx} - iJ \int_x^\infty dy e^{-i\lambda J(y-x)} V(y) F_l(y,\lambda),$$
$$F_r(x,\lambda) = e^{i\lambda Jx} + iJ \int_{-\infty}^x dy e^{-i\lambda J(y-x)} V(y) F_r(y,\lambda).$$

$a_l(\lambda)$ and $a_r(\lambda)$ are known as transition matrices and

$$F_l(x,\lambda) a_r(\lambda) = F_r(x,\lambda), \qquad F_r(x,\lambda) a_l(\lambda) = F_l(x,\lambda),$$

$$a_l(\lambda) = a_r(\lambda)^{-1} = a_r(\lambda)^\dagger$$

Scattering matrix

It is convenient to adopt the following notation:

$$F_l(x,\lambda) = \begin{pmatrix} F_{l1}(x,\lambda) & F_{l2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{l4}(x,\lambda) \end{pmatrix}, \; F_r(x,\lambda) = \begin{pmatrix} F_{r1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{r3}(x,\lambda) & F_{r4}(x,\lambda) \end{pmatrix},$$

where the blocks in *ROSSO* are analytic in $\mathbb{C}^+$ while the *BLU* ones are analytic in $\mathbb{C}^-$. Putting:

$$f_+(x,\lambda) = \begin{pmatrix} F_{l1}(x,\lambda) & F_{r2}(x,\lambda) \\ F_{l3}(x,\lambda) & F_{r4}(x,\lambda) \end{pmatrix}, \; f_-(x,\lambda) = \begin{pmatrix} F_{r1}(x,\lambda) & F_{l2}(x,\lambda) \\ F_{r3}(x,\lambda) & F_{l4}(x,\lambda) \end{pmatrix},$$

we arrive at the following Riemann-Hilbert problem

$$f_-(x,\lambda) = f_+(x,\lambda) J \mathbf{S}(\lambda) J.$$

The scattering matrix $\mathbf{S}(\lambda)$ is J-unitary in the focusing case, i.e., $S(\lambda) = J\left[S(\lambda)^\dagger\right]^{-1} J$:

$$\mathbf{S}(\lambda) = \begin{pmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{pmatrix}.$$

Inverse Scattering Problem

To (re)-construct the potentials we follow these steps:

1. Given the scattering data $\{R(\lambda, t),\, \lambda_j,\, \Gamma_j(t)\}$, let us consider the integral kernel:

$$\Omega(\alpha, t) = \hat{R}(\alpha, t) + \sum_{j=1}^{N} \Gamma_{lj}(t)\, e^{-\lambda_j \alpha}$$

Inverse Scattering Problem

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$$\Omega(\alpha, t) = \hat{R}(\alpha, t) + \sum_{j=1}^{N} \Gamma_{lj}(t)\, e^{-\lambda_j \alpha}$$

2. By using $\Omega(\alpha, t)$, we write down the following integral Marchenko equation

$$\begin{aligned} B_l(x, \alpha, t) &= -\Omega(\alpha + 2x, t) \\ &\quad - \int_0^\infty d\beta B_l(x, \beta, t) \int_0^\infty d\gamma\, \Omega(\gamma + \beta + 2x, t)^\dagger\, \Omega(\alpha + \gamma + 2x, t) \end{aligned}$$

Inverse Scattering Problem

To (re)-construct the potentials we follow these steps:

1. Given the scattering data $\{R(\lambda, t),\, \lambda_j,\, \Gamma_j(t)\}$, let us consider the integral kernel:

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3. The potential $u(x, t)$ is related to the solution of the Marchenko equation by the following relation

$$u(x, t) = 2B_l(x, 0^{+}, t).$$

Marchenko integral kernel

For $t = 0$, consider

$$\Omega(y) = C\, e^{-yA} B,$$

where A is a square matrix of order p having only eigenvalues with positive real parts and B, C are rectangular matrices of order $p \times n$ and $m \times p$, respectively. Moreover, (A, B, C) is a so-called minimal triplet, i.e.,

$$\bigcap_{r=1}^{+\infty} \ker CA^{r-1} = \bigcap_{r=1}^{+\infty} \ker B^{\dagger}(A^{\dagger})^{r-1} = \{0\}.$$

A triplet which give a minimal representation for $\Omega(y)$ is unique up a similar transformation $(A, B, C) \to (EAE^{-1}, EB, CE^{-1})$ where E is an invertible matrix.

Marchenko integral kernel

In general, the kernel $\Omega(y;t)$ of the Marchenko equation satisfy the following :

$$\Omega_t - 4i\,\Omega_{yy} = 0.$$

The previous evolution law suggests the following choice

$$\Omega(y;t) = Ce^{-yA}e^{4i\,A^2 t}B.$$

Choosing the Marchenko integral kernel as above indicated, the Marchenko equation is a separable integral equation and then can be explicitly solved.

Explicit solutions of the Marchenko equation

Substituting

$$\Omega(y,t) = C\, e^{-yA+4iA^2 t} B, \quad \Omega(y,t)^\dagger = B^\dagger e^{-A^\dagger y - 4i(A^\dagger)^2 t} C^\dagger,$$

in the Marchenko equation, it becomes

$$B_l(x,\alpha,t) + C\, e^{-(\alpha+2x)A+4iA^2 t} B +$$
$$\int_0^\infty d\beta \int_0^\infty d\gamma\, K(x,\beta,t) B_l^\dagger e^{-A^\dagger(\gamma+\beta+2x)-4i(A^\dagger)^2 t} C^\dagger C\, e^{-(\alpha+\gamma+2x)A+4iA^2 t} B = 0.$$

Introducing the matrices

$$Q = \int_0^\infty d\gamma\, e^{-\gamma A^\dagger} C^\dagger C e^{-\gamma A}, \qquad N = \int_0^\infty d\beta\, e^{-\beta A} B B^\dagger e^{-\beta A^\dagger},$$

and looking for solutions of the form

$$B_l(x,\alpha,t) = H(x,t)\, e^{-A\alpha+4i(A)^2 t} B,$$

we arrive at the solution

$$B_l(x,\alpha,t) = -CG(x,t)^{-1} e^{-A\alpha} B.$$

Explicit solution of the mNLS equation

The matrix $G(x,t)$ which appear in the solution of the Marchenko equation is express in terms of the matrix triplet (A,B,C)

$$G(x,t) = e^{-\beta} + N\, e^{\beta^\dagger} Q,$$

where $\beta = 2Ax + 4iA^2 t$. Moreover, $G(x,t)$ is invertible on the entire plane xt and its inverse matrix satisfy the following properties: $G(x,t)^{-1} \to 0$ decay esponentially for $x \to \pm\infty$ (for each fixed t.)

Recalling that $u(x,t) = 2B_l(x, 0^+, t)$ we get the solution of the mNLS

$$u(x,t) = -2CG(x,t)^{-1}B.$$

This function exists on the entire plane xt (where it is analytic and exponentially decays).

Sine-Gordon Equation

Let us consider the sine-Gordon

$$u_{xt} = \sin u,$$

where $u(x,t)$ is a real function.

This equation appear in many interesting applicative context such as:

- Description of surfaces of constant mean curvature;
- Magnetic flux propagation in Josephson junctions, i.e. gaps between two superconductors;
- Propagation of deformations along the DNA double helix.

Sine-Gordon equation

In this case, we have to solve the Marchenko equation

$$K(x,y,t) - \Omega(y+x,t)^\dagger + \int_x^\infty dv \int_x^\infty dr\, K(x,v,t)\Omega(v+r,t)\Omega(r+y,t)^\dagger = 0.$$

The integral kernel of this equation obeys to the following PDE:

$$\Omega_{yt} = \frac{1}{2}\Omega,$$

which suggest to make the following choice

$$\Omega(y,t) = C\, e^{-yA} e^{-A^{-1}t/2} B$$

Making considerations similar to those which allow us to get explicit solutions to the m(NLS) equation, we can determine explicitly $K(x,y,t)$.

Sine-Gordon equation

Taking into account that $u(x,t)$ is a real function, also $K(x,y,t)$ has this property. These quantities are related from this relationship:

$$u_x(x,t) = 4K(x,x,t).$$

Developing the computations we get the following equivalent formulas:

$$u(x,t) = -4\int_x^\infty dr\, B^\dagger F(r,t)^{-1} C^\dagger = -4\int_x^\infty dr\, C[F(r,t)^\dagger]^{-1} B,$$

where $F(x,t) = e^{\beta^\dagger} + Q\, e^{-\beta} N$ and $\beta = 2Ax + \dfrac{1}{2} A^{-1} t$.

$$u(x,t) = -4\int_x^\infty dr\, CE(r,t)^{-1} B,$$

with $F(x,t)^\dagger = E(x,t) := e^{\beta} + P\, e^{-\beta} P, \quad \text{and } \beta = 2Ax + \dfrac{1}{2} A^{-1} t.$

The admissible class

It is natural to looking for a larger class including triplets more general than those considered in the previous slides. In fact, for every triplet in this new class we want to repeat the procedure illustrated before obtaining explicit solutions of the sine-Gordon equation.

The triplet (A, B, C) of size p belongs to the *admissible class* $\mathcal{A}$ if:

- The matrices A, B, and C are all real valued.
- The triplet (A, B, C) corresponds to a minimal realization for $\Omega(y, t)$.
- None of the eigenvalues of A are purely imaginary and no two eigenvalues of A can occur symmetrically with respect to the imaginary axis in the complex plane.

Admissible class

For any triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ belonging to the admissible class $\mathcal{A}$ the following properties are satisfied:

I. The Lyapunov equations $\tilde{Q}\tilde{A} + \tilde{A}^\dagger \tilde{Q} = \tilde{C}^\dagger \tilde{C}$, $\tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger$ are uniquely solvable, and their solutions (invertibles and selfadjoints) are given by:

$$\tilde{Q} = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I + i\tilde{A}^\dagger)^{-1} \tilde{C}^\dagger \tilde{C} (\lambda I - i\tilde{A})^{-1},$$

$$\tilde{N} = \frac{1}{2\pi} \int_\gamma d\lambda \, (\lambda I - i\tilde{A})^{-1} \tilde{B}\tilde{B}^\dagger (\lambda I + i\tilde{A}^\dagger)^{-1}.$$

II. The resulting matrix

$$\tilde{F}(x,t) = e^{2\tilde{A}^\dagger x + \frac{1}{2}(\tilde{A}^\dagger)^{-1} t} + \tilde{Q}\, e^{-2\tilde{A}^\dagger x - \frac{1}{2}(\tilde{A}^\dagger)^{-1} t} \tilde{N}$$

is real valued and invertible on the entire xt-plane, and the function

$$\tilde{u}(x,t) = -4 \int_x^\infty dr \, B^\dagger \tilde{F}(r,t)^{-1} C^\dagger$$

is a analytic solution to the sine-Gordon equation everywhere on the xt-plane.

Equivalents Triplet

We say that two triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent if they lead to the same potential $u(x, t)$.

A natural question is the following: *Starting from one triplet in the admissible class, is it possible to get an equivalent triplet such that the matrices A, B, C are real, give a minimal representation for the kernel $\Omega(y, t)$ and all the eigenvalues of A have positive real parts?*

The answer is: YES, however...

$$\boxed{\begin{array}{c}(\tilde{A}, \tilde{B}, \tilde{C}) \text{ is in the} \\ \text{admissible class}\end{array}} \xrightarrow[\text{Equivalent Triplets?}]{\text{How construct}} \boxed{\begin{array}{c}(A, B, C) \text{ minimal representation} \\ \text{where } \mathrm{Re}\lambda_j > 0\end{array}}$$

We have the following:
For any admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$, *there corresponds an equivalent admissible triplet* (A, B, C) *in such a way that all eigenvalues of* A *have positive real parts.*

In order to construct the transformation which allow us to pass from the admissible triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ to the final triplet (A, B, C), it is suitable to consider the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ in a partition form:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix},$$

where all eigenvalues of $\tilde{A}_1$ have positive real parts and all eigenvalues of $\tilde{A}_2$ have negative real parts, and the sizes of the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{C}_1$, $\tilde{C}_2$ are $q \times q$, $(p-q) \times (p-q)$, $q \times 1$, $(p-q) \times 1$, $1 \times q$, and $1 \times (p-q)$, respectively.
For the matrices solutions of the corresponding Lyapunov:

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ \tilde{Q}_3 & \tilde{Q}_4 \end{bmatrix}, \qquad \tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_4 \end{bmatrix}.$$

Triplet in “canonical form”

The equivalent triplet (A, B, C) is built as follows:

$$A_1 = \tilde{A}_1, \qquad A_2 = -\tilde{A}_2^\dagger, \qquad B_1 = \tilde{B}_1 - \tilde{N}_2 \tilde{N}_4^{-1} \tilde{B}_2, \qquad B_2 = \tilde{N}_4^{-1} \tilde{B}_2,$$
$$C_1 = \tilde{C}_1 - \tilde{C}_2 \tilde{Q}_4^{-1} \tilde{Q}_3, \qquad C_2 = \tilde{C}_2 \tilde{Q}_4^{-1}.$$

It is possible to refine the construction in such a way that the triplet (A, B, C) is in a particular form:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix},$$

where each A_j is a Jordan block, instead B_j and C_j are in the following form:

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_j := \begin{bmatrix} c_{jn_j} & \cdots & c_{j2} & c_{j1} \end{bmatrix}$$

Actual research topics

- Exact solutions to the Integrable Discrete Nonlinear Schroedinger system (IDNLS system) by using the IST and Marchenko equations (with Cornelis van der Mee).
- Explicit solutions of the focusing/defocusing NLS equations with boundary nonvanishing conditions. (with Barbara Prinari (University of Colorado at Colorado Sprigs and Università del Salento), Cornelis van der Mee and Federica Vitale (Università del Salento)).
- Darboux transformation. (with Tuncay Aktosun (University of Texas at Arlington) and C. van der Mee).
- Su un problema "quasi" integrabile" (Giovanni Ortenzi (Università di Milano-Bicocca))

Sistemi quasi integrabili

Consideriamo la seguente equazione **NLS with damping**

$$iu_t + u_{xx} + 2|u|^2 u = -i\gamma u, \qquad \gamma > 0$$

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