Soliton Solutions of the Heisenberg Ferromagnetic Equation

Francesco Demontis

(joint work with S. Lombardo, M. Sommacal, C. van der Mee, and F. Vargiu)

Università degli Studi di Cagliari
Dipartimento di Matematica e Informatica

WASCOM 2015
Cetraro (CS), June 1-5 2015
To find explicit soliton solutions of the Landau-Lifshitz equation (LL)

\[ m_t = m \wedge m_{zz} + m \wedge Jm \]

where \( m(z, t) \in \mathbb{R}^3 \) is a vector function satisfying some suitable asymptotic conditions and \( J = \text{diag}\{J_1, J_2, J_3\} \) is a real diagonal matrix with \( J_1 \leq J_2 \leq J_3 \) and \( J_1 < J_3 \).

Importance of the above model

- At the nano length scale, it models the magnetization dynamics of a one-dimensional ferromagnetic system in the presence of crystalline anisotropy.
- In 2013 magnetic-droplet solitons have been experimentally observed [Science] at the nano length scale and they are expected to have important applications.

LL is integrable but it is very difficult to find explicit solutions (see, R.F. Bikbaev, A.I. Bobenko, and A.R. Its, Th. Math. Phys., 2014.)
The Continuous Heisenberg Ferromagnetic Chain Equation (HF)

\[ m_t = m \land m_{zz}, \]

where \( m(z, t) \in \mathbb{R}^3 \) with \( \|m(z, t)\| = 1 \), and \( m(z, t) \to e_3 = (0, 0, 1)^T \) as \( z \to \pm\infty \).

At nanometer scale, in the absence of anisotropy and external magnetic field, HF models the magnetization dynamics of one-dimensional ferromagnet.

Main result on HF:

- **1977 L. A. Takhtajan**, Inverse Scattering Transform to the HF.
Main Purposes

1 To construct an explicit soliton solution formula for the HF;
2 To develop rigorously the Inverse Scattering Transform (IST) for the HF.

METHOD: IST + matrix triplet.

ADVANTAGE: A compact solution formula in terms of elementary functions.
IST for HF

\[
\begin{cases}
  m_t = m \land m_{zz}, \\
  m(z, 0) \text{ known}
\end{cases}
\]

LAX PAIR:

\[V_z = AV = (i\lambda(m \cdot \sigma)) V, \quad V_t = BV = (-2\lambda^2(m \cdot \sigma) - \lambda(\tau \cdot \sigma)) V,\]

where \(\tau = m \land m_z\) and \(\sigma\) is the column vector with entries the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\[m(z) \xrightarrow{\text{direct scattering at } t=0} \{R(\lambda, 0), \{i\lambda_j, c_j\}\} \xrightarrow{\text{HF solution}} m(z, t) \xleftarrow{\text{inverse scattering at } t} \{R(\lambda)e^{-4i\lambda_j^2t}, \{i\lambda_j, c_j e^{-4i\lambda_j^2t}\}\} \xrightarrow{\text{time evolution}} \]
Direct scattering problem: Jost solutions

**Jost matrix solutions**

\[
\begin{align*}
\Psi(z, \lambda) &= (\psi(z, \lambda) \quad \overline{\psi}(z, \lambda)) = e^{i\lambda z \sigma_3}[l_2 + o(1)], \quad z \to +\infty, \\
\Phi(z, \lambda) &= (\phi(z, \lambda) \quad \phi(z, \lambda)) = e^{i\lambda z \sigma_3}[l_2 + o(1)], \quad z \to -\infty.
\end{align*}
\]

Putting \( m^0 = m - e_3 \), the Jost solutions have to satisfy the following

\[
\begin{align*}
\Psi(z, \lambda) &= e^{i\lambda z \sigma_3} - i\lambda \int_z^{\infty} d\hat{z} \ e^{-i\lambda(\hat{z}-z)\sigma_3}(m^0(\hat{z}) \cdot \sigma)\Psi(\hat{z}, \lambda), \\
\Phi(z, \lambda) &= e^{i\lambda z \sigma_3} + i\lambda \int_{-\infty}^{z} d\hat{z} \ e^{i\lambda(z-\hat{z})\sigma_3}(m^0(\hat{z}) \cdot \sigma)\Phi(\hat{z}, \lambda).
\end{align*}
\]

\( \Psi(z, \lambda) \) and \( \Phi(z, \lambda) \) belong to the group \( SU(2) \). *RED* denotes analyticity for \( \lambda \) in \( \mathbb{C}^+ \) while *BLUE* analyticity for \( \lambda \) in \( \mathbb{C}^- \) and the \( \dagger \) symbol denotes transpose conjugation.

**Impossible to find asymptotic properties as** \( \lambda \to \infty \).
Suppose that:

a. \( m'(z) \cdot \sigma \) has its entries in \( L^1(\mathbb{R}) \)

b. \( m_3(z) > -1 \) for each \( z \in \mathbb{R} \).

After an integration by parts and some algebraic manipulation, we obtain

\[
C(z)\Psi(z, \lambda) = e^{i\lambda z \sigma_3} - \int_{z}^{\infty} d\hat{z} \ e^{-i\lambda(\hat{z} - z)\sigma_3} C'(\hat{z})\Psi(\hat{z}, \lambda),
\]

\[
C(z)\Phi(z, \lambda) = e^{i\lambda z \sigma_3} + \int_{-\infty}^{z} d\hat{z} \ e^{i\lambda(z - \hat{z})\sigma_3} C'(\hat{z})\Phi(\hat{z}, \lambda)
\]

where \( C(z) = \frac{1}{2} \begin{pmatrix} 1 + m_3 & m_1 - im_2 \\ -m_1 - im_2 & 1 + m_3 \end{pmatrix} \).

Properties of the matrix \( C(z) \): \( \det C(z) = \frac{1}{2}(1 + m_3) \) and \( C(z) \to l_2 \) as \( z \to \pm\infty \).

The two columns of \( \Psi(z, \lambda) \) and \( \Phi(z, \lambda) \) have a finite limit as \( \lambda \to \infty \) from within the closure of its half-plane of analyticity.
Riemann-Hilbert problem

\[ \Psi(z, \lambda) = \Phi(z, \lambda) a_r(\lambda), \quad \lambda \in \mathbb{R}. \]

where

\[ a_r(\lambda) = \begin{pmatrix} a(\lambda) & -b(\lambda) \\ b(\lambda)^* & a(\lambda)^* \end{pmatrix}, \quad \lambda \in \mathbb{R} \]

and \( |a(\lambda)|^2 + |b(\lambda)|^2 = 1 \). We assume that \( a(\lambda) \neq 0 \) for each \( \lambda \in \mathbb{R} \).

\[ F^-(z, \lambda) = (\overline{\phi}(z, \lambda) \quad \overline{\psi}(z, \lambda)) \quad \text{and} \quad F^+(z, \lambda) = (\psi(z, \lambda) \quad \phi(z, \lambda)) \]

\[ F^-(z, \lambda) = F^+(z, \lambda) \sigma_3 S(\lambda) \sigma_3, \]

where the scattering matrix is given by:

\[ S(\lambda) = \begin{pmatrix} T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda) \end{pmatrix}. \]

\[ S(\lambda)^\dagger = \sigma_3 S(\lambda)^{-1} \sigma_3, \quad \lambda \in \mathbb{R}. \]
Towards the Inverse Scattering Problem

\[ \Psi(z, \lambda) = H(z)e^{i\lambda z\sigma_3} + \int_z^\infty d\hat{z} K(z, \hat{z})e^{i\lambda \hat{z}\sigma_3}, \]

where \( H(z) \) satisfies \( H(z) = \sigma_2 H(z)^* \sigma_2 \) and \( H(z) \to I_2 \) as \( z \to +\infty \), while the auxiliary matrix function

\[ K(z, \lambda) = \begin{pmatrix} K_1(z, \lambda) & -K_2(z, \lambda)^* \\ K_2(z, \lambda) & K_1(z, \lambda)^* \end{pmatrix}, \]

where \( K_1(z, \lambda) \) and \( K_2(z, \lambda) \) are scalar functions.

Putting \( K_\bullet(z) = \int_z^\infty dy \ K(z, y) \), we have

\[ I_2 = \Psi(z, 0) = H(z) + K_\bullet(z), \]

A similar representation holds also for \( \Phi(z, \lambda) \).
**Inverse Scattering Problem: Marchenko equations**

\( \mathbf{K}(x, y) \) has to satisfy the Marchenko integral equations

\[
\mathbf{K}(x, y) + \mathbf{H}(z)\mathbf{F}(z + y) + \int_{z}^{\infty} du \mathbf{K}(z, u)\mathbf{F}(u + y) = 0_{2 \times 2},
\]

where

\[
\mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^{*} & 0 \end{pmatrix}, \quad \mathbf{F}(w) = \rho(w) + \sum_{s=1}^{N} e^{-\lambda_{s}w} c_{s}.
\]

Putting \( \tilde{\mathbf{K}}(z, y) = \mathbf{H}(z)\tilde{\mathbf{K}}(z, y) \), the Marchenko equation becomes

\[
\tilde{\mathbf{K}}(z, y) + \mathbf{F}(z + y) + \int_{z}^{\infty} du \tilde{\mathbf{K}}(z, u)\mathbf{F}(u + y) = 0_{2 \times 2}.
\]

Since \( \tilde{\mathbf{K}}_{\bullet}(z) = \mathbf{H}(z)^{-1}\mathbf{K}_{\bullet}(z) \), we also have

\[
l_{2} = \mathbf{H}(z) + \mathbf{K}_{\bullet}(z) = \mathbf{H}(z) \left\{ l_{2} + \tilde{\mathbf{K}}_{\bullet}(z) \right\} \quad \Longrightarrow \quad \mathbf{H}(z) = \left\{ l_{2} + \tilde{\mathbf{K}}_{\bullet}(z) \right\}^{-1}
\]
Solutions of the HF equations

1. Given the scattering data \( \{ R(\lambda), \lambda_j, c_j \} \), build the function

\[
F(w) = \rho(w) + \sum_{s=1}^{N} e^{-\lambda_s w} c_s
\]

2. Construct \( \mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix} \)

3. Solve the Marchenko equation

\[
\tilde{K}(z, y) + \mathbf{F}(z + y) + \int_{z}^{\infty} du \tilde{K}(z, u)\mathbf{F}(u + y) = 0_{2 \times 2}
\]

where \( y > z \).

4. The solution \( \mathbf{m}(z, t) \) of the HF is given by:

\[
\mathbf{m}(z) \cdot \sigma = \mathbf{H}(z)\sigma_3 \mathbf{H}(z)^{-1} = \left[ I_{2} + \tilde{K}(z) \right]^{-1} \sigma_3 \left[ I_{2} + \tilde{K}(z) \right].
\]
The soliton solutions are characterized by the condition $R(\lambda) = 0$. In this case

$$F(w) = \sum_{s=1}^{N} e^{-\lambda_s w} c_s = Ce^{-sA}B,$$

where $A$ is a square matrix, $B$ is a column vector and $C$ is a row vector. All the eigenvalues of $A$ have positive real parts.

The Marchenko’s kernel can be written as:

$$F(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix} = Ce^{-wA}B,$$

where $A = \begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A^\dagger \end{pmatrix}$, $B = \begin{pmatrix} 0_{n\times 1} & B \\ -C^\dagger & 0_{n\times 1} \end{pmatrix}$, $C = \begin{pmatrix} C & 0_{1\times n} \\ 0_{1\times n} & B^\dagger \end{pmatrix}$.

For such kernels the Marchenko equation can be solved explicitly by separation of variables.
Explicit soliton solutions

Looking for a solution in the form

\[ \tilde{K}(z, y) = -L(z)e^{-yA}B, \]

we get

\[ \tilde{K}_\bullet(z) = -Ce^{-zA} \left[ l_{2n} + e^{-zA}P e^{-zA} \right]^{-1} e^{-yA}A^{-1}B, \]

where

\[ P = \begin{pmatrix} 0_{q \times q} & N \\ Q & 0_{q \times q} \end{pmatrix}, \quad N = \int_0^\infty dz \ e^{-zA}BB^\dagger e^{-zA}, \quad Q = \int_0^\infty dz \ e^{-zA^\dagger} C^\dagger Ce^{-zA}. \]

Time evolution:

\[ (A, B, C) \mapsto \left( A, B, C e^{-4itA^2} \right). \]

HF solution:

\[ m(z, t) \cdot \sigma = \left[ l_2 + \tilde{K}_\bullet(z, t) \right]^{-1} \sigma_3 \left[ l_2 + \tilde{K}_\bullet(z, t) \right]. \]
One soliton-solution

$A = (a)$, with $p = \{\text{Re}\} (a) > 0$, $B = (1)$ and $C = (c)$ with $c \equiv c(t) = ke^{-4ia^2t}$, $k \in \mathbb{C}$ and $k \neq 0$.

The explicit one-soliton solution is given by:

$m_1 (z) + im_2 (z) = \mu (z), \quad m_3 (z) = 1 - \frac{2p^2/|a|^2}{\cosh^2 [2p(z)]},$

where

$\mu (z) = -2 \frac{c^*}{a^*} \frac{e^{-2a^*z}}{1 + (|c|^2/4p^2)e^{-4pz}} \left[ 1 - \frac{|c|^2}{2pa^*} \frac{e^{-4pz}}{1 + (|c|^2/4p^2)e^{-4pz}} \right].$

It is usual to express the one soliton-solution in terms of the velocity $v$ along the $x$-axis and the precessional frequency $\omega$.

$a = p + iq$ with $p = \frac{1}{2} \sqrt{4\omega - v^2}$, $q = \frac{v}{4}$ and $k = 2p = \sqrt{4\omega - v^2}$. 
Numerical examples: One soliton-solution

Figura: $m_1$

Figura: $m_2$

Figura: $m_3$

Figura: $m_3; t = 5$
Numerical examples: Two soliton-solution

Figura: $m_1$

Figura: $m_2$

Figura: $m_3$

Figura: $m_3$: $t=0$, $t=10$, $t=15$
Outlook

We plan to extend, by following the same methodology, this study to the more general Landau-Lifshitz equation

$$m_t = m \wedge m_{zz} + m \wedge Jm$$

where $m(z, t) \in \mathbb{R}^3$ is a vector function satisfying some suitable asymptotic conditions and $J = \text{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix with $J_1 \leq J_2 \leq J_3$ and $J_1 < J_3$. 

References


Thank you!!