Soliton Solutions of the Heisenberg Ferromagnetic Equation

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SOMMACAL's CHALLENGE

To find explicit soliton solutions of the **Landau-Lifshitz** equation (**LL**)

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz} + \mathbf{m} \wedge J\mathbf{m}$$

where $\mathbf{m}(z, t) \in \mathbb{R}^3$ is a vector function satisfying some suitable asymptotic conditions and $J = \mathrm{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix with $J_1 \leq J_2 \leq J_3$ and $J_1 < J_3$.

Importance of the above model

a. At the nano length scale, it models the magnetization dynamics of a one-dimensional ferromagnetic system in the presence of crystalline anisotropy.

b. In 2013 magnetic-droplet solitons have been experimentally observed [**Science**] at the nano length scale and they are expected to have important applications.

LL is integrable but it is very difficult to find explicit solutions (see, R.F. Bikbaev, A.I. Bobenko, and A.R. Its, Th. Math. Phys., 2014.)

The Continuous Heisenberg Ferromagnetic Chain Equation (HF)

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz} \,,$$

where $\mathbf{m}(z,t) \in \mathbb{R}^3$ with $\|\mathbf{m}(z,t)\| = 1$, and $\mathbf{m}\,(z,t) \to \mathbf{e}_3 = (0,0,1)^T$ as $z \to \pm\infty$.

At nanometer scale, in the absence of anisotropy and external magnetic field, HF models the magnetization dynamics of one-dimensional ferromagnet.

Main result on HF:

- 1977 J. Tjon and J. Wright, soliton solutions.
- 1977 L. A. Takhtajan, Inverse Scattering Transform to the HF.
- 1979 V. F. Zakharov and L. A. Takhtajan, gauge equivalence of NLS and Heisenberg equation.

Main Purposes

1. To construct an **explicit soliton solution formula** for the HF;
2. To develop rigorously the **I**nverse **S**cattering **T**ransform (**IST**) for the HF.

METHOD: IST+ matrix triplet.

ADVANTAGE: A compact solution formula in terms of elementary functions.

IST for HF

$$\begin{cases} \mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz} \,, \\ \mathbf{m}(z,0) \ \text{known} \end{cases}$$

LAX PAIR:

$$V_z = AV = (i\lambda(\mathbf{m}\cdot\boldsymbol{\sigma}))\,V\,, \quad V_t = BV = \left(-2\lambda^2(\mathbf{m}\cdot\boldsymbol{\sigma}) - \lambda(\boldsymbol{\tau}\cdot\boldsymbol{\sigma})\right)\,V,$$

where $\boldsymbol{\tau} = \mathbf{m} \wedge \mathbf{m}_z$ and $\boldsymbol{\sigma}$ is the column vector with entries the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\begin{array}{ccc} \mathbf{m}(z) & \xrightarrow{\text{direct scattering at } t=0} & \{R(\lambda,0), \{i\lambda_j, c_j\}\} \\ \Big\downarrow \text{HF solution} & & \Big\downarrow \text{time evolution} \\ \mathbf{m}(z,t) & \xleftarrow[\text{inverse scattering at } t]{} & \left\{R(\lambda)e^{-4i\lambda_j^2 t}, \{i\lambda_j, c_j\, e^{-4i\lambda_j^2 t}\}\right\} \end{array}$$

Direct scattering problem: Jost solutions

Jost matrix solutions

$$\Psi(z,\lambda) = \begin{pmatrix}\psi(z,\lambda) & \overline{\psi}(z,\lambda)\end{pmatrix} = e^{i\lambda z\sigma_3}[I_2 + o(1)], \qquad z \to +\infty\,,$$
$$\Phi(z,\lambda) = \begin{pmatrix}\overline{\phi}(z,\lambda) & \phi(z,\lambda)\end{pmatrix} = e^{i\lambda z\sigma_3}[I_2 + o(1)], \qquad z \to -\infty.$$

Putting $\mathbf{m}^0 = \mathbf{m} - \mathbf{e}_3$, the Jost solutions have to satisfy the following

$$\Psi(z,\lambda) = e^{i\lambda z\sigma_3} - i\lambda \int_z^\infty d\hat{z}\, e^{-i\lambda(\hat{z}-z)\sigma_3}(\mathbf{m}^0(\hat{z})\cdot\boldsymbol{\sigma})\Psi(\hat{z},\lambda),$$
$$\Phi(z,\lambda) = e^{i\lambda z\sigma_3} + i\lambda \int_{-\infty}^z d\hat{z}\, e^{i\lambda(z-\hat{z})\sigma_3}(\mathbf{m}^0(\hat{z})\cdot\boldsymbol{\sigma})\Phi(\hat{z},\lambda).$$

$\Psi(z,\lambda)$ and $\Phi(z,\lambda)$ belong to the group $SU(2)$. *RED* denotes analyticity for λ in $\mathbb{C}^+$ while *BLUE* analyticity for λ in $\mathbb{C}^-$ and the $\dagger$ symbol denotes transpose conjugation.

Impossible to find asympotic properties as $\lambda \to \infty$.

Direct scattering problem: Jost solutions

Suppose that:

a. $\mathbf{m}'(z) \cdot \boldsymbol{\sigma}$ has its entries in $L^1(\mathbb{R})$

b. $m_3(z) > -1$ for each $z \in \mathbb{R}$.

After an integration by parts and some algebraic manipulation, we obtain

$$C(z)\Psi(z,\lambda) = e^{i\lambda z\sigma_3} - \int_z^\infty d\hat{z}\, e^{-i\lambda(\hat{z}-z)\sigma_3} C'(\hat{z})\Psi(\hat{z},\lambda),$$
$$C(z)\Phi(z,\lambda) = e^{i\lambda z\sigma_3} + \int_{-\infty}^z d\hat{z}\, e^{i\lambda(z-\hat{z})\sigma_3} C'(\hat{z})\Phi(\hat{z},\lambda)$$

where $C(z) = \frac{1}{2}\begin{pmatrix} 1+m_3 & m_1 - im_2 \\ -m_1 - im_2 & 1+m_3 \end{pmatrix}$.

Properties of the matrix $C(z)$: $\det C(z) = \frac{1}{2}(1+m_3)$ and $C(z) \to I_2$ as $z \to \pm\infty$.

The two columns of $\Psi(z,\lambda)$ and $\Phi(z,\lambda)$ have a finite limit as $\lambda \to \infty$ from within the closure of its half-plane of analyticity

Riemann-Hilbert problem

$$\Psi(z,\lambda) = \Phi(z,\lambda)a_r(\lambda), \qquad \lambda \in \mathbb{R}.$$

where

$$a_r(\lambda) = \begin{pmatrix} a(\lambda) & -b(\lambda) \\ b(\lambda)^* & a(\lambda)^* \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

and $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$. We assume that $a(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$.

$\mathbf{F}_-(z,\lambda) = \begin{pmatrix} \overline{\phi}(z,\lambda) & \overline{\psi}(z,\lambda) \end{pmatrix}$ and $\mathbf{F}_+(z,\lambda) = \begin{pmatrix} \psi(z,\lambda) & \phi(z,\lambda) \end{pmatrix}$

$$\mathbf{F}_-(z,\lambda) = \mathbf{F}_+(z,\lambda)\sigma_3 S(\lambda)\sigma_3,$$

where the *scattering matrix* is given by:

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda) \end{pmatrix}.$$

$$S(\lambda)^\dagger = \sigma_3 S(\lambda)^{-1}\sigma_3, \qquad \lambda \in \mathbb{R}.$$

Towards the Inverse Scattering Problem

$$\Psi(z,\lambda) = \mathbf{H}(z)e^{i\lambda z\sigma_3} + \int_z^\infty d\hat{z}\,\mathbf{K}(z,\hat{z})e^{i\lambda\hat{z}\sigma_3},$$

where $\mathbf{H}(z)$ satisfies $\mathbf{H}(z) = \sigma_2\mathbf{H}(z)^*\sigma_2$ and $\mathbf{H}(z) \to I_2$ as $z \to +\infty$, while the auxiliary matrix function

$$\mathbf{K}(z,\lambda) = \begin{pmatrix} \mathbf{K}_1(z,\lambda) & -\mathbf{K}_2(z,\lambda)^* \\ \mathbf{K}_2(z,\lambda) & \mathbf{K}_1(z,\lambda)^* \end{pmatrix},$$

where $\mathbf{K}_1(z,\lambda)$ and $\mathbf{K}_2(z,\lambda)$ are scalar functions.

Putting $\mathbf{K}_\bullet(z) = \int_z^\infty dy\,\mathbf{K}(z,y)$, we have

$$I_2 = \Psi(z,0) = \mathbf{H}(z) + \mathbf{K}_\bullet(z),$$

A similar representation holds also for $\Phi(z,\lambda)$.

Inverse Scattering Problem: Marchenko equations

$\mathbf{K}(x, y)$ has to satisfy the Marchenko integral equations

$$\mathbf{K}(x, y) + \mathbf{H}(z)\mathbf{F}(z + y) + \int_z^\infty du\, \mathbf{K}(z, u)\mathbf{F}(u + y) = 0_{2\times 2},$$

where

$$\mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix}, \quad F(w) = \rho(w) + \sum_{s=1}^{N} e^{-\lambda_s w} c_s.$$

Putting $\mathbf{K}(z, y) = \mathbf{H}(z)\tilde{\mathbf{K}}(z, y)$, the Marchenko equation becomes

$$\tilde{\mathbf{K}}(z, y) + \mathbf{F}(z + y) + \int_z^\infty du\, \tilde{\mathbf{K}}(z, u)\mathbf{F}(u + y) = 0_{2\times 2}.$$

Since $\tilde{\mathbf{K}}_\bullet(z) = \mathbf{H}(z)^{-1}\mathbf{K}_\bullet(z)$, we also have

$$I_2 = \mathbf{H}(z) + \mathbf{K}_\bullet(z) = \mathbf{H}(z)\left\{I_2 + \tilde{\mathbf{K}}_\bullet(z)\right\} \Longrightarrow \mathbf{H}(z) = \left\{I_2 + \tilde{\mathbf{K}}_\bullet(z)\right\}^{-1}$$

Solutions of the HF equations

1 Given the scattering data $\{R(\lambda),\, \lambda_j,\, c_j\}$, build the function

$$F(w) = \rho(w) + \sum_{s=1}^{N} e^{-\lambda_s w} c_s$$

2 Construct $\mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix}$

3 Solve the Marchenko equation

$$\tilde{\mathbf{K}}(z, y) + \mathbf{F}(z + y) + \int_z^\infty du\, \tilde{\mathbf{K}}(z, u)\mathbf{F}(u + y) = 0_{2\times 2}$$

where $y > z$.

4 The solution $\mathbf{m}(z, t)$ of the HF is given by:

$$\mathbf{m}(z) \cdot \boldsymbol{\sigma} = \mathbf{H}(z)\sigma_3\mathbf{H}(z)^{-1} = \left[I_2 + \tilde{\mathbf{K}}_\bullet(z)\right]^{-1} \sigma_3 \left[I_2 + \tilde{\mathbf{K}}_\bullet(z)\right].$$

Triplet Method

The soliton solutions are characterized by the condition $R(\lambda) = 0$.

In this case

$$F(w) = \sum_{s=1}^{N} e^{-\lambda_s w} c_s = Ce^{-sA}B,$$

where A is a square matrix, B is a column vector and C is a row vector. All the eigenvalues of A have positive real parts.

The Marchenko's kernel can be written as:

$$\mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix} = \mathcal{C}e^{-w\mathcal{A}}\mathcal{B},$$

where $\mathcal{A} = \begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A^\dagger \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 0_{n\times 1} & B \\ -C^\dagger & 0_{n\times 1} \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} C & 0_{1\times n} \\ 0_{1\times n} & B^\dagger \end{pmatrix}$.

For such kernels the Marchenko equation can be solved explicitely by separation of variables.

Explicit soliton solutions

Looking for a solution in the form

$$\tilde{\mathbf{K}}(z,y) = -\mathbf{L}(z)e^{-y\mathcal{A}}\mathcal{B},$$

we get

$$\tilde{\mathbf{K}}_\bullet(z) = -\mathcal{C}e^{-z\mathcal{A}}\left[I_{2n} + e^{-z\mathcal{A}}\mathcal{P}\, e^{-z\mathcal{A}}\right]^{-1} e^{-y\mathcal{A}}\mathcal{A}^{-1}\mathcal{B},$$

where

$$\mathcal{P} = \begin{pmatrix} 0_{q\times q} & N \\ -Q & 0_{q\times q} \end{pmatrix},\ N = \int_0^\infty dz\, e^{-zA}BB^\dagger e^{-zA^\dagger},\ Q = \int_0^\infty dz\, e^{-zA^\dagger}C^\dagger Ce^{-zA}.$$

Time evolution:

$$(A,B,C) \mapsto \left(A,B,C\,e^{-4itA^2}\right).$$

HF solution:

$$\mathbf{m}(z,t)\cdot\boldsymbol{\sigma} = \left[I_2 + \tilde{\mathbf{K}}_\bullet(z,t)\right]^{-1}\sigma_3\left[I_2 + \tilde{\mathbf{K}}_\bullet(z,t)\right].$$

One soliton-solution

$A = (a)$, with $p = \{Re\}(a) > 0$, $B = (1)$ and $C = (c)$ with $c \equiv c(t) = ke^{-4ia^2 t}$, $k \in \mathbb{C}$ and $k \neq 0$.

The explicit one-soliton solution is given by:

$$m_1(z) + im_2(z) = \mu(z), \quad m_3(z) = 1 - \frac{2p^2/|a|^2}{\cosh^2[2p(z)]},$$

where

$$\mu(z) = -2\frac{c^*}{a^*}\frac{e^{-2a^* z}}{1 + (|c|^2/4p^2)e^{-4pz}}\left[1 - \frac{|c|^2}{2pa^*}\frac{e^{-4pz}}{1 + (|c|^2/4p^2)e^{-4pz}}\right].$$

It is usual to express the one soliton-solution in terms of the velocity v along the x-axis end the precessional frequency ω.

$a = p + iq$ with $p = \frac{1}{2}\sqrt{4\omega - v^2}$, $q = \dfrac{v}{4}$ and $k = 2p = \sqrt{4\omega - v^2}$.

Numerical examples: One soliton-solution

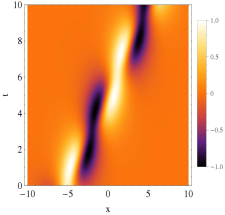

Figura: m_1

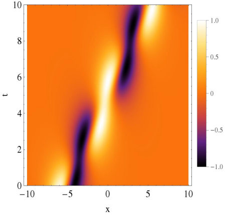

Figura: m_2

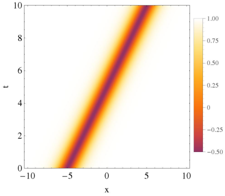

Figura: m_3

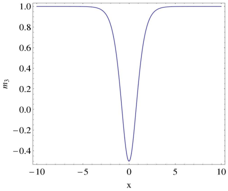

Figura: m_3; $t = 5$

Numerical examples: Two soliton-solution

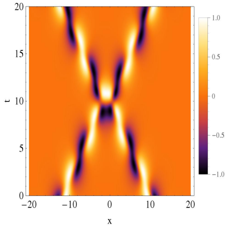

Figura: m_1

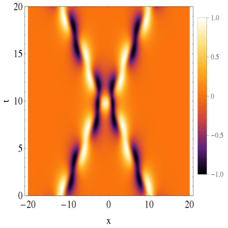

Figura: m_2

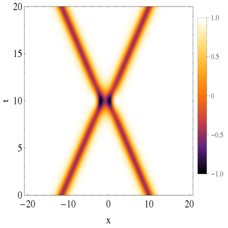

Figura: m_3

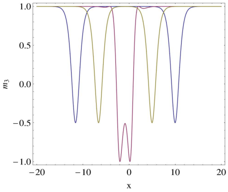

Figura: m_3: t=0, t=10, t=15

Outlook

We plan to extend, by following the same methodology, this study to the more general **Landau-Lifshitz** equation

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz} + \mathbf{m} \wedge J\mathbf{m}$$

where $\mathbf{m}(z, t) \in \mathbb{R}^3$ is a vector function satisfying some suitable asymptotic conditions and $J = \text{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix with $J_1 \leq J_2 \leq J_3$ and $J_1 < J_3$.

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Acknowledgments

Thank you!!