Closed Form Solutions to the Discrete Nonlinear Schrödinger Equation

Francesco Demontis

(based on a joint work with C. van der Mee)

Università degli Studi di Cagliari
Dipartimento di Matematica e Informatica

AGMP-7. Algebra Geometry Mathematical Physics
Mulhouse, 24-26 October 2011

Supported by RAS under grant PO Sardegna 2007-2013, L.R. 7/2007

Contents

a. Introduction to the IDNLS equation

b. IDNLS equation and inverse scattering transform

c. Explicit solutions of the IDNLS equation

d. Continuous matrix NLS limit

e. Examples

Introduction

In this talk we consider the system of the **I**ntegrable **D**iscrete **N**onlinear **S**chrödinger (IDNLS) equations

$$i\frac{d}{d\tau}\mathbf{u}_n = \mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1} - \mathbf{u}_{n+1}\mathbf{w}_n\mathbf{u}_n - \mathbf{u}_n\mathbf{w}_n\mathbf{u}_{n-1},$$
$$-i\frac{d}{d\tau}\mathbf{w}_n = \mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1} - \mathbf{w}_{n+1}\mathbf{u}_n\mathbf{w}_n - \mathbf{w}_n\mathbf{u}_n\mathbf{w}_{n-1},$$

where n is an integer labeling "position" and $\mathbf{u}_n$ and $\mathbf{w}_n$ are $N \times M$ and $M \times N$ matrix functions depending on "time" $\tau \in \mathbb{R}$.

When $\mathbf{w}_n = -\mathbf{u}_n^\dagger$ we have the so-called *focusing case*.

Introduction

Because the IDNLS can be seen as a finite difference approximation of the matrix NLS, it has the same applications of the matrix NLS, i.e.

- electromagnetic wave propagation in nonlinear media;
- surface waves on deep waters;
- signal propagation in optical fibers.

Introduction

Important dates on the IDNLS:

- Ablowitz and Ladik (1975-1976) were the first authors to show that IDNLS can be solved via the IST. In these works they only considered the scalar case.
- Gerdjikov and Ivanov (1981-1984) developed the direct and inverse scattering theory of the discrete Zakharov-Shabat system in the matrix case.
- Tsuchida, Ujino and Wadati (2002) and Ablowitz, Prinari,Trubatch (2004) applied the IST to the matrix IDNLS.

It is worthwhile to remark that also other techniques can be applied to solve the IDNLS. For example, the Hirota method can be used to derive some special solutions (in fact, D. Cai, A.P. Bishop, and N. Grønbech-Jensen did it to obtain the breather solutions to the IDNLS).

Description of Method

Representing the kernels of the Marchenko equation in a separated form by using two triplets of constant matrices, we explicitly solve the Marchenko equations by separation of variables.

Our main result (for *the focusing case*) is as follows:
Starting from two "suitable" triplets of matrices, we get the following explicit solution formula

$$\mathbf{u}_n(\tau) = -2\mathcal{B}^\dagger[{\mathcal{A}^\dagger}^{2(n+1)} e^{i\tau(\mathcal{A}^\dagger - {\mathcal{A}^\dagger}^{-1})^2} + 4\mathbf{Q}\mathcal{A}^{-2(n+2)} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathbf{N}{\mathcal{A}^\dagger}^{-2}]^{-1}\mathcal{C}^\dagger,$$

where

$$\mathbf{Q} = \sum_{\sigma=0}^{\infty} {\mathcal{A}^\dagger}^{-2\sigma}\mathcal{C}^\dagger\mathcal{C}\mathcal{A}^{-2\sigma}, \qquad \mathbf{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma}\mathcal{B}\mathcal{B}^\dagger{\mathcal{A}^\dagger}^{-2\sigma}.$$

Advantages

In particular, we get the following advantages:

- We obtain a compact solution formula in terms of two triplets of matrices. These solutions can be unzipped as a compound of polynomial, exponential and trigonometric functions by using Mathematica or Matlab.
- Choosing the triplets in an appropriate way, we can see that our formula contains the N-soliton and breather solutions as special cases (but also the multipole solutions).
- Our procedure also allows us to overcome the assumption (made in order to obtain appropriate symmetry properties of its scattering data)

$$\mathbf{u}_n \mathbf{w}_n = \mathbf{w}_n \mathbf{u}_n = c_n I_N, \qquad n \in \mathbb{Z}, \tag{1}$$

where $\{1 - c_n\}_{n=-\infty}^{\infty}$ is a sequence of nonzero scalars.

Discrete Zakharov-Shabat System

In order to solve the initial-value problem for the IDNLS by using the inverse scattering transform (IST) method, it is necessary to build the direct and inverse scattering for the following so-called matrix discrete Zakharov-Shabat (Z-S) system

$$\mathbf{v}_{n+1} = \begin{pmatrix} zI_N & \mathbf{u}_n \\ \mathbf{w}_n & z^{-1}I_M \end{pmatrix} \mathbf{v}_n,$$

where z is the (complex) spectral parameter and $I_N - \mathbf{u}_n\mathbf{w}_n$ is assumed nonsingular for each $n \in \mathbb{Z}$ (which is always true in the focusing case) and the potentials $\{\mathbf{u}_n\}_{n=-\infty}^{\infty}$ and $\{\mathbf{w}_n\}_{n=-\infty}^{\infty}$ satisfy the ℓ^1-condition

$$\sum_{n=-\infty}^{\infty} \{\|\mathbf{u}_n\| + \|\mathbf{w}_n\|\} < +\infty,$$

and $\|\cdot\|$ denotes any matrix norm.
We do not assume the *quasiscalarity assumption* (which implies $N = M$ and is always satisfied for $N = M = 1$)

$$\mathbf{u}_n\mathbf{w}_n = \mathbf{w}_n\mathbf{u}_n = c_nI_N, \qquad n \in \mathbb{Z},$$

where $\{1 - c_n\}_{n=-\infty}^{\infty}$ is a sequence of nonzero complex numbers.

Direct and Inverse Scattering of the Zakharov-Shabat System

We recall that

THE DIRECT SCATTERING consists of: Determine the reflection coefficients, the bound states [poles of $\mathbf{t}_r(\lambda)$], and norming constants from the potentials $\mathbf{u}_n, \mathbf{w}_n$.

THE INVERSE SCATTERING consists of: Reconstruct the potentials $\mathbf{u}_n, \mathbf{w}_n$ from one reflection coefficient, the bound states, and the norming constants.

Direct Scattering: Jost Solutions

Let us define the four *Jost solutions* $\phi_n(z)$, $\bar{\phi}_n(z)$, $\psi_n(z)$, and $\bar{\psi}_n(z)$ as those $(N+M)\times N$, $(N+M)\times M$, $(N+M)\times M$, and $(N+M)\times N$ matrix solutions to the discrete Z-S system

$$\phi_n(z) \sim z^n \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \qquad \bar{\phi}_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \qquad n \to -\infty,$$
$$\psi_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \quad \bar{\psi}_n(z) \sim z^n \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \qquad n \to +\infty.$$

Jost Solutions

We can consider the $(N+M) \times (N+M)$ matrices

$$\begin{pmatrix} \phi_n(z) & \bar{\phi}_n(z) \end{pmatrix}, \qquad \begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix}.$$

We have used two colors because $z^n\psi_n(z)$ and $z^{-n}\phi_n(z)$ are continuous in $|z| \geq 1$ and are analytic in $|z| > 1$ (RED), while $z^{-n}\bar{\psi}_n(z)$ and $z^n\bar{\phi}_n(z)$ are continuous in $|z| \leq 1$ and analytic in $|z| < 1$ (BLUE).

We have

$$\begin{pmatrix} \phi_n(z) & \bar{\phi}_n(z) \end{pmatrix} = \begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix} \mathbf{T}(z),$$
$$\begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \bar{\phi}_n(z) \end{pmatrix} \bar{\mathbf{T}}(z),$$

where $I_N - \mathbf{u}_n\mathbf{w}_n$ (and hence $I_M - \mathbf{w}_n\mathbf{u}_n$) is nonsingular for each $n \in \mathbb{Z}$ and

$$\mathbf{T}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}(z) & \bar{\mathbf{b}}(z) \\ \mathbf{b}(z) & \bar{\mathbf{a}}(z) \end{pmatrix}, \qquad \bar{\mathbf{T}}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{\mathbf{c}}(z) & \mathbf{d}(z) \\ \bar{\mathbf{d}}(z) & \mathbf{c}(z) \end{pmatrix},$$

are called the *transition coefficient matrices*.

Scattering matrices

Using the analiticity properties of the Jost solutions, we arrive at the following Riemann-Hilbert problem

$$\begin{pmatrix} \bar{\psi}_n(z) & \bar{\phi}_n(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} J\mathbf{S}(z)J, \qquad |z| = 1,$$
$$\begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} = \begin{pmatrix} \bar{\psi}_n(z) & \bar{\phi}_n(z) \end{pmatrix} J\bar{\mathbf{S}}(z)J, \qquad |z| = 1,$$

where $J = I_N \oplus (-I_M)$

$$\mathbf{S}(z) = \begin{pmatrix} \mathbf{t}_r(z) & \boldsymbol{\ell}(z) \\ \boldsymbol{\rho}(z) & \mathbf{t}_l(z) \end{pmatrix}, \qquad \bar{\mathbf{S}}(z) = \mathbf{S}(z)^{-1} = \begin{pmatrix} \bar{\mathbf{t}}_l(z) & \bar{\boldsymbol{\rho}}(z) \\ \bar{\boldsymbol{\ell}}(z) & \bar{\mathbf{t}}_r(z) \end{pmatrix}.$$

are called the *scattering matrices*. Morever, if there are no spectral singularities

$$\boldsymbol{\rho}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\boldsymbol{\rho}}(s), \qquad \bar{\boldsymbol{\rho}}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\bar{\boldsymbol{\rho}}}(s),$$
$$\bar{\boldsymbol{\ell}}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\bar{\boldsymbol{\ell}}}(s), \qquad \boldsymbol{\ell}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\boldsymbol{\ell}}(s).$$

Inverse Scattering Problem

In order to (re)-construct the potential we have the following procedure:

1. Given the scattering data $\{\boldsymbol{\ell}(z), \zeta_k, \mathbf{C}_j\}$ and $\{\bar{\boldsymbol{\ell}}(z), \bar{\zeta}_k, \bar{\mathbf{C}}_j\}$, we build the following kernels

$$\mathbf{F}(j) = \hat{\boldsymbol{\ell}}(j) + \sum_k \zeta_k^{-(j+1)} \mathbf{C}_k, \quad \bar{\mathbf{F}}(j) = \hat{\bar{\boldsymbol{\ell}}}(j) - \sum_k \bar{\zeta}_k^{-j-1} \bar{\mathbf{C}}_k.$$

Inverse Scattering Problem

In order to (re)-construct the potential we have the following procedure:

1. Given the scattering data $\{\boldsymbol{\ell}(z),\, \zeta_k,\, \mathbf{C}_j\}$ and $\{\bar{\boldsymbol{\ell}}(z),\, \bar{\zeta}_k,\, \bar{\mathbf{C}}_j\}$, we build the following kernels

$$\mathbf{F}(j) = \hat{\boldsymbol{\ell}}(j) + \sum_k \zeta_k^{-(j+1)} \mathbf{C}_k, \quad \bar{\mathbf{F}}(j) = \hat{\bar{\boldsymbol{\ell}}}(j) - \sum_k \bar{\zeta}_k^{j-1} \bar{\mathbf{C}}_k.$$

2. Using the function $\mathbf{F}(j)$ and $\bar{\mathbf{F}}(j)$, we can consider the following Marchenko equation

$$\bar{L}^{\mathrm{up}}(n, n-2\sigma-1) = -\mathbf{F}(2[n-\sigma]-1) + \sum_{\sigma'=0}^{\infty} \bar{L}^{\mathrm{up}}(n, n-2\sigma'-1) \sum_{\sigma''=1}^{\infty} \bar{\mathbf{F}}(2[n-\sigma'-\sigma'']-1)\mathbf{F}(2[n-\sigma''-\sigma]-1),$$

Inverse Scattering Problem

In order to (re)-construct the potential we have the following procedure:

1. Given the scattering data $\{\boldsymbol{\ell}(z), \zeta_k, \mathbf{C}_j\}$ and $\{\bar{\boldsymbol{\ell}}(z), \bar{\zeta}_k, \bar{\mathbf{C}}_j\}$, we build the following kernels
$$\mathbf{F}(j) = \hat{\boldsymbol{\ell}}(j) + \sum_k \zeta_k^{-(j+1)}\mathbf{C}_k, \quad \bar{\mathbf{F}}(j) = \hat{\bar{\boldsymbol{\ell}}}(j) - \sum_k \bar{\zeta}_k^{j-1}\bar{\mathbf{C}}_k.$$
2. Using the function $\mathbf{F}(j)$ and $\bar{\mathbf{F}}(j)$, we can consider the following Marchenko equation
$$\bar{L}^{\mathrm{up}}(n, n-2\sigma-1) = -\mathbf{F}(2[n-\sigma]-1)$$
$$+\sum_{\sigma'=0}^{\infty} \bar{L}^{\mathrm{up}}(n, n-2\sigma'-1)\sum_{\sigma''=1}^{\infty} \bar{\mathbf{F}}(2[n-\sigma'-\sigma'']-1)\mathbf{F}(2[n-\sigma''-\sigma]-1),$$
3. The potential $\mathbf{u}_n$ ($\mathbf{w}_n$ can be derived by considering a similar Marchenko equation) are connected to the above equations by means of the following relationship
$$\mathbf{u}_n = -\bar{L}^{\mathrm{up}}(n+1, n).$$

Remarks

Remark 1: The relation

$$\mathbf{u}_n = -\overline{L}^{\text{up}}(n+1, n),$$

holds also if the quasiscalarity condition $(\mathbf{u}_n \mathbf{w}_n = \mathbf{w}_n \mathbf{u}_n = c_n I_N,\ n \in \mathbb{Z},)$ is NOT satisfied.

Remark 2: It is well known that the discrete spectrum of the discrete Z-S system is invariant under sign inversion. Then we have

- the transmission coefficients are even functions of z.
- the reflection coefficients are odd functions of z.
- the functions $\hat{\boldsymbol{\ell}}(s)$, $\hat{\bar{\boldsymbol{\ell}}}(s)$ (as well as $\hat{\boldsymbol{\rho}}(s)$, $\hat{\bar{\boldsymbol{\rho}}}(s)$) vanish if s is even.
- the Marchenko kernels $\mathbf{F}(j)$ and $\bar{\mathbf{F}}(j)$ vanish if j is even.

Kernels in terms of the matrix triplets

Taking into account the properties of the kernels $\mathbf{F}_l(j)$ and $\bar{\mathbf{F}}_l(j)$, we assume the following representation for these kernels

$$\mathbf{F}(j) = \hat{\ell}(j) + [1 + (-1)^{j+1}]\mathcal{C}\mathcal{A}^{-(j+1)}\mathcal{B},$$
$$\bar{\mathbf{F}}(j) = \hat{\bar{\ell}}(j) + [1 + (-1)^{j+1}]\bar{\mathcal{C}}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}},$$

where the triplets (A, B, C), $(\bar{A}, \bar{B}, \bar{C})$ are in the following form:

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A} \end{pmatrix}, \quad B = \begin{pmatrix} \mathcal{B} \\ \mathcal{B} \end{pmatrix}, \quad C = \begin{pmatrix} \mathcal{C} & \mathcal{C} \end{pmatrix},$$
$$\bar{A} = \begin{pmatrix} \bar{\mathcal{A}} & 0 \\ 0 & -\bar{\mathcal{A}} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{\mathcal{B}} \\ \bar{\mathcal{B}} \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{\mathcal{C}} & \bar{\mathcal{C}} \end{pmatrix}.$$

Moreover, these triplets have to satisfy the following properties:

- A has only eigenvalues of modulus larger than one.
- $\bar{A}$ is nonsingular matrix which has only eigenvalues of modulus less than one.

Explicit Solutions to the Marchenko Equations

Substituting the two expressions of the kernels into the Marchenko equation, we get

$$\overline{L}^{\text{up}}(n, n-2\sigma-1) = -2\mathcal{C}\mathcal{A}^{2[n-\sigma-1]}\mathcal{B}$$
$$+4\sum_{\sigma'=0}^{\infty}\overline{L}^{\text{up}}(n, n-2\sigma-1)\sum_{\sigma''=1}^{\infty}\bar{\mathcal{C}}\bar{\mathcal{A}}^{-2[n-\sigma'-\sigma'']}\bar{\mathcal{B}}\mathcal{C}\mathcal{A}^{2[n-\sigma''-\sigma-1]}\mathcal{B}.$$

If we look for a solution in the following form

$$\overline{L}^{\text{up}}(n, n-2\sigma-1) = H(n)\mathcal{A}^{2[n-\sigma-2]}\mathcal{B}$$

and define

$$\mathcal{Q} = \sum_{\sigma=0}^{\infty}\bar{\mathcal{A}}^{2\sigma}\bar{\mathcal{B}}\mathcal{C}\mathcal{A}^{-2\sigma}, \qquad \mathcal{N} = \sum_{\sigma=0}^{\infty}\mathcal{A}^{-2\sigma}\mathcal{B}\bar{\mathcal{C}}\bar{\mathcal{A}}^{2\sigma},$$

we obtain

$$H(n) = -2\mathcal{C}\mathcal{A}^{2}[I - 4\mathcal{A}^{2(n-2)}\mathcal{N}\bar{\mathcal{A}}^{-2(n-1)}\mathcal{N}]^{-1}.$$

Solutions of IDNLS

Since

$$\overline{L}^{\mathrm{up}}(n, n-2\sigma-1) = -2\mathcal{C}\mathcal{A}^2[I - 4\mathcal{A}^{2(n-2)}\mathcal{N}\bar{\mathcal{A}}^{-2(n-1)}\mathcal{N}]^{-1}\mathcal{A}^{2[n-\sigma-2]}\mathcal{B},$$

and recalling that $\mathbf{u}_n = -\overline{L}^{\mathrm{up}}(n+1, n)$, we get

$$\mathbf{u}_n = -2\mathcal{C}[\mathcal{A}^{-2n} - 4\mathcal{N}\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}.$$

Starting from the equation

$$L^{\mathrm{dn}}(n, n-2\sigma-1) = -\bar{\mathbf{F}}(2[n-\sigma]-1)$$
$$+\sum_{\sigma'=0}^{\infty} L^{\mathrm{dn}}(n, n-2\sigma'-1)\sum_{\sigma''=1}^{\infty}\mathbf{F}(2[n-\sigma'-\sigma'']-1)\bar{\mathbf{F}}(2[n-\sigma''-\sigma]-1)$$

whose solution is related to IDNLS solution by $\mathbf{w}_n = -L^{\mathrm{dn}}(n+1, n)$, and proceeding in the (essentially) same way we get

$$\mathbf{w}_n = -2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{2(n+1)} - 4\mathcal{Q}\mathcal{A}^{2(n-1)}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{B}}.$$

The time dependence

The final step to obtain the solutions of the IDNLS system consists of considering the time dependence of the scattering data. When the reflection coefficients vanish, the kernels must satisfy the following

$$
\begin{aligned}
i\frac{d}{d\tau}\mathbf{F}(n;\tau) &= \mathbf{F}(n+2;\tau) - 2\mathbf{F}(n;\tau) + \mathbf{F}(n-2;\tau),\\
-i\frac{d}{d\tau}\bar{\mathbf{F}}(n;\tau) &= \bar{\mathbf{F}}(n+2;\tau) - 2\bar{\mathbf{F}}(n;\tau) + \bar{\mathbf{F}}(n-2;\tau),
\end{aligned}
$$

and hence the kernels depending on time can be chosen as

$$
\begin{aligned}
\mathbf{F}(j;\tau) &= 2\mathcal{C}\mathcal{A}^{-(j+1)}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B},\\
\bar{\mathbf{F}}(j;\tau) &= 2\bar{\mathcal{C}}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}}.
\end{aligned}
$$

CRUCIAL OBSERVATION:

Let us consider the kernels

$$\mathbf{F}(j) = 2\mathcal{C}\mathcal{A}^{-(j+1)}\mathcal{B},$$
$$\bar{\mathbf{F}}(j) = 2\bar{\mathcal{C}}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}}.$$

Making the following changes

$$\mathcal{B} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B}, \quad \bar{\mathcal{C}} \mapsto \bar{\mathcal{C}}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},$$

and leaving unchanged $\mathcal{A}$, $\mathcal{C}$, $\bar{\mathcal{B}}$, we get

$$\mathbf{F}(j;\tau) = 2\mathcal{C}\mathcal{A}^{-(j+1)}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B},$$
$$\bar{\mathbf{F}}(j;\tau) = 2\bar{\mathcal{C}}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}\bar{\mathcal{A}}^{j-1}\bar{\mathcal{B}},$$

which are the kernels depending on time. Under the substitutions above indicated we also have

$$\mathcal{N} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{N}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}$$

while $\mathcal{Q}$ remains unchanged.

Solutions of the IDNLS

Then executing the substitutions

$$\mathcal{B} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{B}, \quad \bar{\mathcal{C}} \mapsto \bar{\mathcal{C}}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},$$
$$\mathcal{N} \mapsto e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{N}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},$$

and leaving unchanged $\mathcal{A}$, $\mathcal{C}$, $\bar{\mathcal{B}}$, and $\mathcal{Q}$ into the expression

$$\mathbf{u}_n = -2\mathcal{C}[\mathcal{A}^{-2n} - 4\mathcal{N}\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}$$
$$\mathbf{w}_n = -2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{2(n+1)} - 4\mathcal{Q}\mathcal{A}^{2(n-1)}\mathcal{N}\bar{\mathcal{A}}^{2}]^{-1}\bar{\mathcal{B}}$$

we get the solution to the IDNLS equation, i.e.

$$\mathbf{u}_n(\tau) = -2\mathcal{C}[\mathcal{A}^{-2n}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} - 4\mathcal{N}e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B},$$

$$\mathbf{w}_n(\tau) = -2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{2(n+1)}e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2} - 4\mathcal{Q}\mathcal{A}^{2(n-1)}e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2}\mathcal{N}\bar{\mathcal{A}}^{2}]^{-1}\bar{\mathcal{B}}.$$

The focusing case

Relating the matrix triplets in the following way:

$$\bar{\mathcal{A}} = {\mathcal{A}^\dagger}^{-1}, \qquad \bar{\mathcal{B}} = {\mathcal{A}^\dagger}^{-1}\mathcal{C}^\dagger, \qquad \bar{\mathcal{C}} = -\mathcal{B}^\dagger {\mathcal{A}^\dagger}^{-1},$$

the Marchenko kernels satisfy the relation

$$\bar{\mathbf{F}}(j) = -\mathbf{F}(j)^\dagger,$$

which characterize the focusing case. Introducing the quantity

$$\mathbf{Q} = \sum_{\sigma=0}^{\infty} {\mathcal{A}^\dagger}^{-2\sigma}\mathcal{C}^\dagger\mathcal{C}\mathcal{A}^{-2\sigma}, \qquad \mathbf{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma}\mathcal{B}\mathcal{B}^\dagger {\mathcal{A}^\dagger}^{-2\sigma},$$

we can written the general solutions before found as

$$\mathbf{u}_n(\tau) = -2\mathcal{C}[\mathcal{A}^{-2n}e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} + 4\mathbf{N}{\mathcal{A}^\dagger}^{2(n-1)}e^{i\tau(\mathcal{A}^\dagger-{\mathcal{A}^\dagger}^{-1})^2}\mathbf{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}.$$

Moreover, we can easily verify that

$$\mathbf{w}_n(\tau) = -\mathbf{u}_n(\tau)^\dagger, \qquad n \in \mathbb{Z}.$$

Continuous Matrix Limit

We note that

$$i\frac{d}{dt}\mathbf{U}_n = \frac{\mathbf{U}_{n+1} - 2\mathbf{U}_n + \mathbf{U}_{n-1}}{h^2} - \mathbf{U}_{n+1}\mathbf{W}_n\mathbf{U}_n - \mathbf{U}_n\mathbf{W}_n\mathbf{U}_{n-1},$$
$$-i\frac{d}{dt}\mathbf{W}_n = \frac{\mathbf{W}_{n+1} - 2\mathbf{W}_n + \mathbf{W}_{n-1}}{h^2} - \mathbf{W}_{n+1}\mathbf{U}_n\mathbf{W}_n - \mathbf{W}_n\mathbf{U}_n\mathbf{W}_{n-1},$$

is the discretization of the continuous matrix NLS system

$$i\mathbf{U}_t = \mathbf{U}_{xx} - 2\mathbf{UWU},$$
$$-i\mathbf{W}_t = \mathbf{W}_{xx} - 2\mathbf{WUW},$$

obtained by using the following finite differencing scheme:

$$\mathbf{U}_n(t) = \mathbf{U}(nh, t), \qquad \mathbf{W}_n(t) = \mathbf{W}(nh, t).$$

The system with unknowns $\mathbf{U}_n, \mathbf{W}_n$ reduce to the "our" IDNLS system (i.e. the system independent from h and having $\mathbf{u}_n, \mathbf{w}_n$ as unknowns) if we make the following rescaling:

$$\mathbf{u}_n = h\mathbf{U}_n, \qquad \mathbf{w}_n = h\mathbf{W}_n, \qquad \tau = (t/h^2).$$

Continuous Matrix Limit

By using our solution formula (**in the focusing case**) and taking into account that $\tau = (t/h^2)$, we can write

$$\mathbf{U}_n = \frac{\mathbf{u}_n}{h} = -\frac{2}{h}\mathcal{C}[\mathcal{A}^{-2n} e^{ih^{-2}t(\mathcal{A}-\mathcal{A}^{-1})^2} + 4\mathbf{N}\mathcal{A}^{\dagger\,2(n-1)} e^{ih^{-2}t(\mathcal{A}^{\dagger}-\mathcal{A}^{\dagger\,-1})^2}\mathbf{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}.$$

Let us now consider the following rescale of the matrix triplet $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as follows:

$$\mathcal{A} \mapsto e^{h\mathbf{A}}, \qquad \mathcal{B} \mapsto \sqrt{h}\,\mathbf{B}, \qquad \mathcal{C} \mapsto \sqrt{h}\mathbf{C}.$$

First we observe that

$$e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mapsto e^{4it\left(\frac{\sinh(h\mathbf{A})}{h}\right)^2} \sim e^{4it\mathbf{A}^2} \text{ as } h \to 0^+.$$

To see the effect of the rescaling on the formula for $\mathbf{U}_n$ we have to understand how $\mathbf{Q}$ and $\mathbf{N}$ change.

Continuous Matrix Limit

Under the rescaling before introduced, for the matrix **Q** we obtain

$$\mathbf{Q} = \sum_{j=0}^{\infty} \mathcal{A}^{\dagger^{-2j}} \mathcal{C}^{\dagger}\mathcal{C}\mathcal{A}^{-2j} \mapsto h\sum_{j=0}^{\infty} e^{-2jh\mathbf{A}^{\dagger}}\mathbf{C}^{\dagger}\mathbf{C}e^{-2jh\mathbf{A}} \simeq \int_0^{\infty} dz\, e^{-2z\mathbf{A}^{\dagger}}\mathbf{C}^{\dagger}\mathbf{C}e^{-2z\mathbf{A}} = \frac{1}{2}Q$$

where $Q = \int_0^{\infty} dz\, e^{-z\mathbf{A}^{\dagger}}\mathbf{C}^{\dagger}\mathbf{C}e^{-z\mathbf{A}}$ and we have used the trapezoid rule

$$\int_0^{\infty} dx\, F(x) \sim h\{\frac{1}{2}F(0) + \sum_{j=1}^{\infty} F(jh)\}.$$

Making very similar calculations for the matrix **N**, we get $\mathbf{N} = \frac{1}{2}N$ where

$$N = \int_0^{\infty} dz\, e^{-z\mathbf{A}}\mathbf{B}\mathbf{B}^{\dagger}e^{-z\mathbf{A}^{\dagger}}.$$

Continuous Matrix Limit

Finally we can write the expression of $\mathbf{U}_n$ under the rescaling of the triplets. We have

$$\mathbf{U}_n(t) = 2\mathbf{C}[e^{-2nh\mathbf{A}} e^{4it\left(\frac{\sinh(h\mathbf{A})}{h}\right)^2} + 4\frac{N}{2} e^{2nh\mathbf{A}^\dagger} e^{-2h\mathbf{A}^\dagger} e^{4it\left(\frac{\sinh(h\mathbf{A}^\dagger)}{h}\right)^2} \frac{Q}{2} e^{-2h\mathbf{A}}]^{-1}\mathbf{B}\,,$$

and if we calculate the limit of the preceding equation as the stepsize h goes to zero while $x = nh$, we get

$$\mathbf{U}_n \sim -2\mathbf{C}\left[e^{-2x\mathbf{A}} e^{4it\mathbf{A}^2} + Ne^{2x\mathbf{A}^\dagger} e^{4it\mathbf{A}^{\dagger 2}} Q\right]^{-1}\mathbf{B}\,.$$

But this is the solution formula for the mNLS equation obtained by using the triplet method!

An example

In the focusing case we have the following relationships between the triplets

$$\bar{\mathcal{A}} = {\mathcal{A}^{\dagger}}^{-1}, \qquad \bar{\mathcal{B}} = {\mathcal{A}^{\dagger}}^{-1}\mathcal{C}^{\dagger}, \qquad \bar{\mathcal{C}} = -\mathcal{B}^{\dagger}{\mathcal{A}^{\dagger}}^{-1}.$$

If we choose the triplets as $\mathcal{A} = \begin{pmatrix} 3 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 2 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} 1 & 1 \end{pmatrix}$, the previous relations hold. The solutions of the corresponding Stein equations are

$$\mathbf{Q} = \begin{pmatrix} \frac{81}{40} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \frac{81}{20} \end{pmatrix}.$$

We easily calculate (by hand or by using computer algebra) the potentials $\mathbf{u}_n(\tau)$ and $\mathbf{w}_n(\tau)$ via formulas before found, obtaining

$$\mathbf{u}_n(\tau) = \begin{pmatrix} \dfrac{-4}{3^{-2n}e^{\frac{64}{9}it} + \frac{1}{200}3^{4+2n}e^{\frac{64}{9}it}} \\ \dfrac{-4}{3^{-2n}e^{\frac{64}{9}it} + \frac{1}{200}3^{4+2n}e^{\frac{64}{9}it}} \end{pmatrix}$$

$$\mathbf{w}_n(\tau) = \begin{pmatrix} \dfrac{4}{3^{-2n}e^{\frac{64}{9}it} + \frac{1}{200}3^{4+2n}e^{\frac{64}{9}it}} & \dfrac{4}{3^{-2j}e^{-\frac{64}{9}it} + \frac{1}{200}3^{4+2n}e^{-\frac{64}{9}it}} \end{pmatrix}.$$

Example: Plots of the Rectangular Solutions

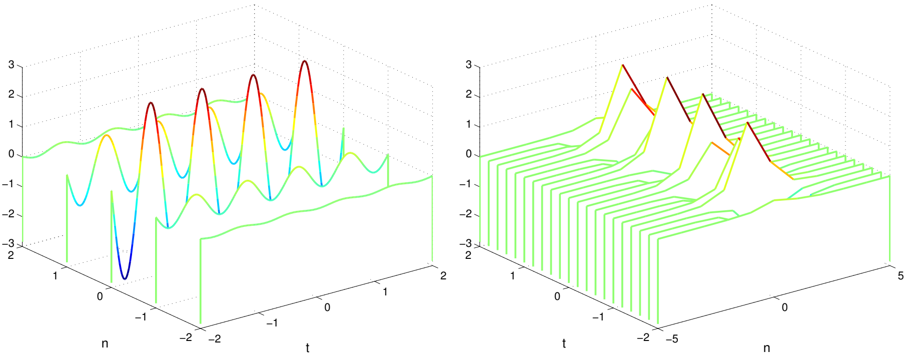

Figura: Real part of the first component of $\mathbf{u}_n(\tau)$ for n fixed (on the left) and t fixed (on the right).

I'm greatly indebted to Antonio Aricò and Giuseppe Rodriguez for their assistance in developing the Mathematica code.

LITERATURE:

- M.J. Ablowitz and J.F. Ladik, *Nonlinear differential-difference equations*, J. Math. Phys. **16**, 598–603 (1975).
- M.J. Ablowitz and J.F. Ladik, *A nonlinear difference scheme and inverse scattering*, Stud. Appl. Math. **55**, 213–229 (1976).
- M.J. Ablowitz and J.F. Ladik, *Nonlinear differential-difference equations and nonlinearity*, J. Math. Phys. **17**, 1011–1018 (1976).
- M.J. Ablowitz, B. Prinari, and A.D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems*, Cambridge University Press, Cambridge, 2004.

LITERATURE: II

- V.S. Gerdjikov and M.I. Ivanov, *Block discrete Zakharov-Shabat system. I. Generalized Fourier expansion*, United Nuclear Research Institute Report E2-81-811, Dubna, 1981.
- V.S. Gerdjikov and M.I. Ivanov, *Block discrete Zakharov-Shabat system. I. Hamiltonian structures*, United Nuclear Research Institute Report E2-81-812, Dubna, 1981.
- V.S. Gerdjikov, M.I. Ivanov, and P.P. Kulish, *Expansions of the "squared solutions and difference evolution equations*, J. Math. Phys. **25**, 25–34 (1984).

LITERATURE: III

- T. Tsuchida, *Integrable discretizations of derivative nonlinear Schrödinger equations*, J. Phys. A **35**, 7827–7847 (2002).
- T. Tsuchida, H. Ujino, and M. Wadati, *Integrable semi-discretization of the coupled modified KdV equations*, J. Math. Phys. **39**, 4785–4813 (1998).
- T. Tsuchida, H. Ujino, and M. Wadati, *Integrable semi-discretization of the coupled nonlinear Schrödinger equations*, J. Phys. A **32**, 2239–2262 (1999).
- D. Cai, A.P. Bishop, and N. Grønbech-Jensen, *Perturbation theories of a discrete, integrable nonlinear Schrödinger equation*, Phys. Rev. E **53**, 1202–1205 (1996).

LITERATURE: IV

- T. Aktosun, F. Demontis, and C. van der Mee, *Exact solutions to the sine-Gordon equation*, Journal of Mathematical Physics, **51**, 123521 (2010).
- F. Demontis and C. van der Mee, *Explicit solutions of the cubic matrix nonlinear Schrödinger Equation*, Inverse Problems **24**, 02520 (2008).
- C. Schiebold, *Solutions of the sine-Gordon equation coming in clusters*, Revista Matemática Complutense **15**, 265-325 (2002).
- H. Dym, *Linear Algebra in Action*, Graduate Studies in Mathematics **78**, Amer. Math. Soc., Providence, RI, 2007.
- F. Demontis and Cornelis van der Mee, *Exact Solutions to the Integrable Discrete Nonlinear Schrödinger Equation under a Quasiscalarity Condition*, Accepted by Communications in Applied and Industrial Mathematics, (2011).

Thank you for your attemption!!!

$\mathcal{Q}$, $\mathcal{N}$ satisfy uniquely the following (matrix) equation

$$\mathcal{Q} - \bar{\mathcal{A}}^2 \mathcal{Q} \mathcal{A}^{-2} = \bar{\mathcal{B}} \mathcal{C}, \qquad \mathcal{N} - \mathcal{A}^{-2} \mathcal{N} \bar{\mathcal{A}}^2 = \mathcal{B} \bar{\mathcal{C}},$$

Q and **N** are the unique solutions of the Stein equations

$$\mathbf{Q} - {\mathcal{A}^\dagger}^{-2} \mathbf{Q} \mathcal{A}^{-2} = \mathcal{C}^\dagger \mathcal{C},$$
$$\mathbf{N} - \mathcal{A}^{-2} \mathbf{N} {\mathcal{A}^\dagger}^{-2} = \mathcal{B} \mathcal{B}^\dagger.$$

$$\mathcal{Q} = {\mathcal{A}^\dagger}^{-1} \mathbf{Q}, \qquad \mathcal{N} = -\mathbf{N} {\mathcal{A}^\dagger}^{-1}.$$