



Closed-form soliton solutions for the Heisenberg Ferromagnetic Chain Equation

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My research interests: Nonlinear “Integrable” PDE (**NIPDE**) in $1 + 1$ variable (continuous and discrete), explicit solutions of such equations

Main Collaborations:

- C. van der Mee and F. Vargiu (University of Cagliari)
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- G. Ortenzi (University of Bergamo)
- S. Lombardo and M. Sommacal (Northumbria University, Newcastle, UK)
- T. Aktosun (University of Texas at Arlington)
- B. Prinari (University of Colorado at Colorado Springs)
- (University of Cagliari)

Integrability

What do I mean for **integrable equation**?

- They admit **soliton** solution
- An algorithm, called **Inverse Scattering Transform (IST)**, can be applied to solve the initial value problem associated to the given NIPDE
- They can be written as completely integrable Hamiltonian systems

In **RED** the definition used in this talk.

We need to know:

- a. (**One million dollar question**:) When is it possible to apply the IST?
- b. How does the IST work?

Towards the IST

Fourier Transform (Linear) \longleftrightarrow IST (Nonlinear)

Solution of linear evolution equation via Fourier transform

Let us consider the following initial value problem

$$\begin{cases} u_t(x, t) = -i\omega(-i\frac{\partial}{\partial x})u(x, t), \\ u(x, 0) = u(x), \quad -\infty < x < \infty \\ \lim_{x \rightarrow \pm\infty} [|x|^{1+\epsilon} u_0(x)] = 0, \quad \epsilon > 0. \end{cases}$$

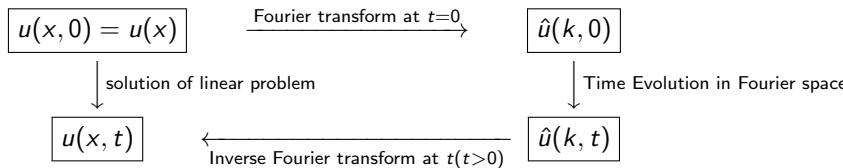
$$\hat{u}(k, t) = \int_{-\infty}^{\infty} dx e^{-ikx} u(x, t),$$
$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \hat{u}(k, t),$$

If $u(x, t)$ satisfies the linear evolution equation,

$$\hat{u}_t(k, t) = -i\omega(k)\hat{u}(k, t) \implies \hat{u}(k, t) = \hat{u}(k, 0)e^{-i\omega(k)t}.$$

Towards the IST

$$\begin{cases} u_t(x, t) = -i\omega(-i\frac{\partial}{\partial x})u(x, t), \\ u(x, 0) = u(x), \quad -\infty < x < \infty \end{cases}$$



By following **an analogous scheme** the IST allows to solve the initial value problem

$$\begin{cases} NLPDE, \\ u(x, 0) = u(x), \end{cases}$$

where $u(x, t)$ is the unknown.

Examples of integrable equations

$$u_t + 6uu_x + u_{xxx} = 0, \quad \text{Korteweg-de Vries equation}$$

$$u_{xt} = \sin u, \quad \text{sine-Gordon equation}$$

$$i u_t + u_{xx} \pm 2uu^\dagger u = 0, \quad \text{Nonlinear Schrödinger (NLS) equation}$$

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz}, \quad \text{The Continuous Heisenberg Ferromagnetic Chain Equation}$$

Linear problems corresponding to the initial value problem associated with the NIPDE :

$$\begin{cases} V_x &= AV, \\ V_t &= BV \end{cases}$$

The zero-curvature condition $A_t - B_x + AB - BA = 0$ “gives” the NIPDE.

ONE EXAMPLE:

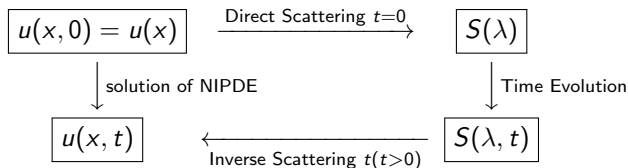
$$\begin{cases} iu_t + u_{xx} \pm 2|u|^2 u = 0 \\ u(x, 0) = u(x) \end{cases}$$

$$A = -i\lambda\sigma_3 + Q, \quad B = -2i\lambda^2\sigma_3 - i\sigma_3 Q^2 + 2\lambda Q + i\sigma_3 Q_x,$$

where

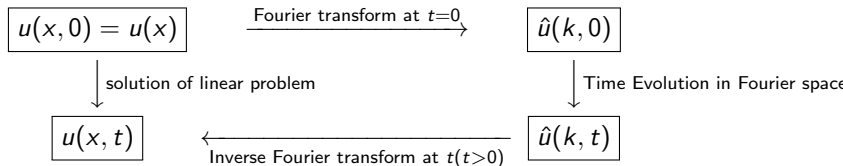
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u(x) \\ \mp u^*(x) & 0 \end{pmatrix}.$$

1. **Direct Scattering Problem:** Given $u(x, 0)$, construct the so-called **scattering data at $t = 0$** by the linear eigenvalue problem $V_x = AV$
2. **Evolution of the Scattering Data:** By using the linear problem $V_t = BV$, determine **the scattering data at time t**
3. **Inverse Scattering Problem:** Known the evolved scattering data, get the function $u(x, t)$ which corresponds to such data

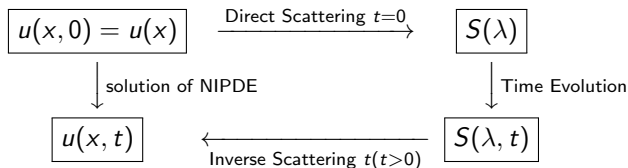


Fourier Transform VS IST

Fourier Transform:



IST:



Explicit solutions and Inverse Scattering Problem

- 1 Given $\{R(\lambda, t), \lambda_j, \Gamma_j(t)\}$, build

$$\Omega(\alpha, t) = \hat{R}(\alpha, t) + \sum_{j=1}^N \Gamma_{lj}(t) e^{-\lambda_j \alpha}$$

Explicit solutions and Inverse Scattering Problem

- 1 Given $\{R(\lambda, t), \lambda_j, \Gamma_j(t)\}$, build

$$\Omega(\alpha, t) = \hat{R}(\alpha, t) + \sum_{j=1}^N \Gamma_{lj}(t) e^{-\lambda_j \alpha}$$

- 2 $\Omega(\alpha, t)$ is the kernel of the following Marchenko integral equation

$$B_I(x, \alpha, t) = -\Omega(\alpha + 2x, t) \\ - \int_0^\infty d\beta B_I(x, \beta, t) \int_0^\infty d\gamma \Omega(\gamma + \beta + 2x, t)^\dagger \Omega(\alpha + \gamma + 2x, t)$$

Explicit solutions and Inverse Scattering Problem

- 1 Given $\{R(\lambda, t), \lambda_j, \Gamma_j(t)\}$, build

$$\Omega(\alpha, t) = \hat{R}(\alpha, t) + \sum_{j=1}^N \Gamma_{lj}(t) e^{-\lambda_j \alpha}$$

- 2 $\Omega(\alpha, t)$ is the kernel of the following Marchenko integral equation

$$B_l(x, \alpha, t) = -\Omega(\alpha + 2x, t) - \int_0^\infty d\beta B_l(x, \beta, t) \int_0^\infty d\gamma \Omega(\gamma + \beta + 2x, t)^\dagger \Omega(\alpha + \gamma + 2x, t)$$

- 3 Determine how $u(x, t)$ is related to the solution of the above equation. Such relationship is different for each NIPDE.

$$u(x, t) = 2B_l(x, 0^+, t), \quad \text{NLS.}$$

SOMMACAL's CHALLENGE

The **Landau-Lifshitz** equation (**LL**)

$$m_t = m \wedge m_{zz} + m \wedge Jm$$

where $m(z, t) \in \mathbb{R}^3$ is a vector function satisfying some suitable asymptotic conditions and $J = \text{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix with $J_1 \leq J_2 \leq J_3$ and $J_1 < J_3$.

Importance of the above model

- At the nano length scale, it models the magnetization dynamics of a one-dimensional ferromagnetic system in the presence of crystalline anisotropy.
- In 2013 **magnetic-droplet solitons** have been experimentally observed [**Science**] at the nano length scale and they are expected to have important applications.

LL is integrable but it is very difficult to find explicit solutions (see, R.F. Bikbaev, A.I. Bobenko, and A.R. Its, Th. Math. Phys., 2014.)

The Continuous Heisenberg Ferromagnetic Chain Equation (HF)

$$\mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz},$$

where $\mathbf{m}(z, t) \in \mathbb{R}^3$ with $\|\mathbf{m}(z, t)\| = 1$, and $\mathbf{m}(z, t) \rightarrow \mathbf{e}_3 = (0, 0, 1)^T$ as $z \rightarrow \pm\infty$.

At nanometer scale, in the absence of anisotropy and external magnetic field, HF models the magnetization dynamics of one-dimensional ferromagnet.

Main result on HF:

- 1977 J. Tjon and J. Wright, soliton solutions.
- 1977 L. A. Takhtajan, Inverse Scattering Transform to the HF.
- 1979 V. F. Zakharov and L. A. Takhtajan, gauge equivalence of NLS and Heisenberg equation.

Pre-print: F. Demontis, S. Lombardo, M. Sommacal, C. van der Mee, F. Vargiu, *Exact Solutions of the Heisenberg Ferromagnetic Equation*

Main Purposes

- 1 To construct an **explicit soliton solution formula** for the HF;
- 2 To develop rigorously the **Inverse Scattering Transform (IST)** for the HF.

METHOD: IST+ matrix triplet.

ADVANTAGE: A compact solution formula in terms of elementary functions.

Main result (?!)

Triplet (A, B, C) of matrices as input: A is a square matrix, B is a column vector and C is a row vector.

$$\mathcal{A} = \begin{pmatrix} A & 0_{n \times n} \\ 0_{n \times n} & A^\dagger \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0_{n \times 1} & B \\ -C^\dagger & 0_{n \times 1} \end{pmatrix}, \mathcal{C} = \begin{pmatrix} C & 0_{1 \times n} \\ 0_{1 \times n} & B^\dagger \end{pmatrix}.$$

$$\mathbf{m}(z) \cdot \boldsymbol{\sigma} = \left[I_2 + \tilde{\mathbf{K}}_\bullet(z) \right]^{-1} \sigma_3 \left[I_2 + \tilde{\mathbf{K}}_\bullet(z) \right],$$

$$\tilde{\mathbf{K}}_\bullet(z) = -C e^{-zA} \left[I_{2n} + e^{-zA} \mathcal{P} e^{-zA} \right]^{-1} e^{-yA} A^{-1} B,$$

where

$$\mathcal{P} = \begin{pmatrix} 0_{n \times n} & N \\ -Q & 0_{n \times n} \end{pmatrix}, N = \int_0^\infty dz e^{-zA} B B^\dagger e^{-zA^\dagger}, Q = \int_0^\infty dz e^{-zA^\dagger} C^\dagger C e^{-zA}.$$

$$(A, B, C) \mapsto (A, B, C e^{-4itA^2}).$$

Example: One soliton-solution

$A = (a)$, with $p = \{Re\}(a) > 0$, $B = (1)$ and $C = (c)$ with $c \equiv c(t) = ke^{-4ia^2t}$, $k \in \mathbb{C}$ and $k \neq 0$.

The explicit **one-soliton solution** is given by:

$$m_1(z) + im_2(z) = \mu(z), \quad m_3(z) = 1 - \frac{2p^2/|a|^2}{\cosh^2[2p(z)]},$$

where

$$\mu(z) = -2\frac{c^*}{a^*} \frac{e^{-2a^*z}}{1 + (|c|^2/4p^2)e^{-4pz}} \left[1 - \frac{|c|^2}{2pa^*} \frac{e^{-4pz}}{1 + (|c|^2/4p^2)e^{-4pz}} \right].$$

It is usual to express the one soliton-solution in terms of the **velocity** v along the x -axis and the **precessional frequency** ω .

$$a = p + iq \text{ with } p = \frac{1}{2}\sqrt{4\omega - v^2}, \quad q = \frac{v}{4} \text{ and } k = 2p = \sqrt{4\omega - v^2}.$$

Numerical examples: One soliton-solution

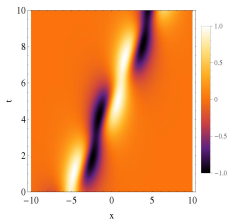


Figura: m_1

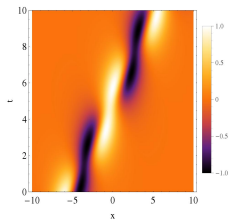


Figura: m_2

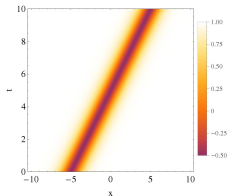


Figura: m_3

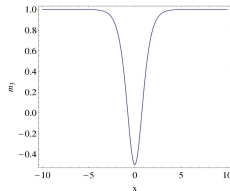


Figura: $m_3; t = 5$

Numerical examples: Two soliton-solution

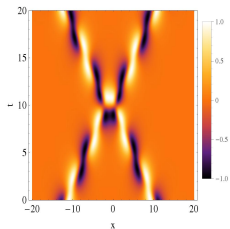


Figura: m_1

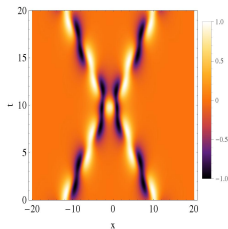


Figura: m_2

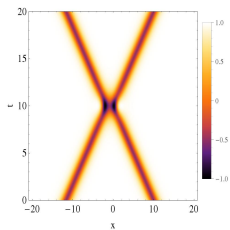


Figura: m_3

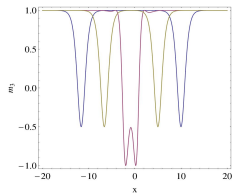


Figura m_3 : $t=0$, $t=10$, $t=15$

$$\begin{cases} \mathbf{m}_t = \mathbf{m} \wedge \mathbf{m}_{zz}, \\ \mathbf{m}(z, 0) \text{ known} \end{cases}$$

LAX PAIR:

$$V_z = AV = (i\lambda(\mathbf{m} \cdot \boldsymbol{\sigma})) V, \quad V_t = BV = (-2\lambda^2(\mathbf{m} \cdot \boldsymbol{\sigma}) - \lambda(\boldsymbol{\tau} \cdot \boldsymbol{\sigma})) V,$$

where $\boldsymbol{\tau} = \mathbf{m} \wedge \mathbf{m}_z$ and $\boldsymbol{\sigma}$ is the column vector with entries the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\begin{array}{ccc} \mathbf{m}(z) & \xrightarrow{\text{direct scattering at } t=0} & \{R(\lambda, 0), \{i\lambda_j, c_j\}\} \\ \downarrow \text{HF solution} & & \downarrow \text{time evolution} \\ \mathbf{m}(z, t) & \xleftarrow{\text{inverse scattering at } t} & \{R(\lambda)e^{-4i\lambda_j^2 t}, \{i\lambda_j, c_j e^{-4i\lambda_j^2 t}\}\} \end{array}$$

Direct scattering problem: Jost solutions

Jost matrix solutions

$$\begin{aligned}\Psi(z, \lambda) &= \begin{pmatrix} \psi(z, \lambda) & \bar{\psi}(z, \lambda) \end{pmatrix} = e^{i\lambda z \sigma_3} [I_2 + o(1)], & z \rightarrow +\infty, \\ \Phi(z, \lambda) &= \begin{pmatrix} \bar{\phi}(z, \lambda) & \phi(z, \lambda) \end{pmatrix} = e^{i\lambda z \sigma_3} [I_2 + o(1)], & z \rightarrow -\infty.\end{aligned}$$

Putting $\mathbf{m}^0 = \mathbf{m} - \mathbf{e}_3$, the Jost solutions have to satisfy the following

$$\begin{aligned}\Psi(z, \lambda) &= e^{i\lambda z \sigma_3} - i\lambda \int_z^\infty d\hat{z} e^{-i\lambda(\hat{z}-z)\sigma_3} (\mathbf{m}^0(\hat{z}) \cdot \boldsymbol{\sigma}) \Psi(\hat{z}, \lambda), \\ \Phi(z, \lambda) &= e^{i\lambda z \sigma_3} + i\lambda \int_{-\infty}^z d\hat{z} e^{i\lambda(z-\hat{z})\sigma_3} (\mathbf{m}^0(\hat{z}) \cdot \boldsymbol{\sigma}) \Phi(\hat{z}, \lambda).\end{aligned}$$

$\Psi(z, \lambda)$ and $\Phi(z, \lambda)$ belong to the group $SU(2)$. **RED** denotes analyticity for λ in \mathbb{C}^+ while **BLUE** analyticity for λ in \mathbb{C}^- and the \dagger symbol denotes transpose conjugation.

Impossible to find asymptotic properties as $\lambda \rightarrow \infty$.

Direct scattering problem: Jost solutions

Suppose that:

- a. $\mathbf{m}'(z) \cdot \boldsymbol{\sigma}$ has its entries in $L^1(\mathbb{R})$
- b. $m_3(z) > -1$ for each $z \in \mathbb{R}$.

After an integration by parts and some algebraic manipulation, we obtain

$$C(z)\Psi(z, \lambda) = e^{i\lambda z\sigma_3} - \int_z^\infty d\hat{z} e^{-i\lambda(\hat{z}-z)\sigma_3} C'(\hat{z})\Psi(\hat{z}, \lambda),$$
$$C(z)\Phi(z, \lambda) = e^{i\lambda z\sigma_3} + \int_{-\infty}^z d\hat{z} e^{i\lambda(z-\hat{z})\sigma_3} C'(\hat{z})\Phi(\hat{z}, \lambda)$$

where $C(z) = \frac{1}{2} \begin{pmatrix} 1 + m_3 & m_1 - im_2 \\ -m_1 - im_2 & 1 + m_3 \end{pmatrix}$.

Properties of the matrix $C(z)$: $\det C(z) = \frac{1}{2}(1 + m_3)$ and $C(z) \rightarrow I_2$ as $z \rightarrow \pm\infty$.

The two columns of $\Psi(z, \lambda)$ and $\Phi(z, \lambda)$ have a finite limit as $\lambda \rightarrow \infty$ from within the closure of its half-plane of analyticity

Riemann-Hilbert problem

$$\Psi(z, \lambda) = \Phi(z, \lambda) a_r(\lambda), \quad \lambda \in \mathbb{R}.$$

where

$$a_r(\lambda) = \begin{pmatrix} a(\lambda) & -b(\lambda) \\ b(\lambda)^* & a(\lambda)^* \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

and $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$. We assume that $a(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$.

$$\mathbf{F}_-(z, \lambda) = \begin{pmatrix} \bar{\phi}(z, \lambda) & \bar{\psi}(z, \lambda) \end{pmatrix} \text{ and } \mathbf{F}_+(z, \lambda) = \begin{pmatrix} \psi(z, \lambda) & \phi(z, \lambda) \end{pmatrix}$$

$$\mathbf{F}_-(z, \lambda) = \mathbf{F}_+(z, \lambda) \sigma_3 S(\lambda) \sigma_3,$$

where the *scattering matrix* is given by:

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R(\lambda) \\ L(\lambda) & T(\lambda) \end{pmatrix}.$$

$$S(\lambda)^\dagger = \sigma_3 S(\lambda)^{-1} \sigma_3, \quad \lambda \in \mathbb{R}.$$

Towards the Inverse Scattering Problem

$$\Psi(z, \lambda) = \mathbf{H}(z)e^{i\lambda z\sigma_3} + \int_z^\infty d\hat{z} \mathbf{K}(z, \hat{z})e^{i\lambda\hat{z}\sigma_3},$$

where $\mathbf{H}(z)$ satisfies $\mathbf{H}(z) = \sigma_2 \mathbf{H}(z)^* \sigma_2$ and $\mathbf{H}(z) \rightarrow I_2$ as $z \rightarrow +\infty$, while the auxiliary matrix function

$$\mathbf{K}(z, \lambda) = \begin{pmatrix} \mathbf{K}_1(z, \lambda) & -\mathbf{K}_2(z, \lambda)^* \\ \mathbf{K}_2(z, \lambda) & \mathbf{K}_1(z, \lambda)^* \end{pmatrix},$$

where $\mathbf{K}_1(z, \lambda)$ and $\mathbf{K}_2(z, \lambda)$ are scalar functions.

Putting $\mathbf{K}_\bullet(z) = \int_z^\infty dy \mathbf{K}(z, y)$, we have

$$I_2 = \Psi(z, 0) = \mathbf{H}(z) + \mathbf{K}_\bullet(z),$$

A similar representation holds also for $\Phi(z, \lambda)$.

Inverse Scattering Problem: Marchenko equations

$K(x, y)$ has to satisfy the **Marchenko integral equations**

$$K(x, y) + H(z)F(z + y) + \int_z^\infty du K(z, u)F(u + y) = 0_{2 \times 2},$$

where

$$F(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix}, \quad F(w) = \rho(w) + \sum_{s=1}^N e^{-\lambda_s w} c_s.$$

Putting $K(z, y) = H(z)\tilde{K}(z, y)$, the Marchenko equation becomes

$$\tilde{K}(z, y) + F(z + y) + \int_z^\infty du \tilde{K}(z, u)F(u + y) = 0_{2 \times 2}.$$

Since $\tilde{K}_\bullet(z) = H(z)^{-1}K_\bullet(z)$, we also have

$$I_2 = H(z) + K_\bullet(z) = H(z) \left\{ I_2 + \tilde{K}_\bullet(z) \right\} \implies H(z) = \left\{ I_2 + \tilde{K}_\bullet(z) \right\}^{-1}$$

Solutions of the HF equations

- 1 Given the scattering data $\{R(\lambda), \lambda_j, c_j\}$, build the function

$$F(w) = \rho(w) + \sum_{s=1}^N e^{-\lambda_s w} c_s$$

- 2 Construct $\mathbf{F}(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix}$

- 3 Solve the Marchenko equation

$$\tilde{\mathbf{K}}(z, y) + \mathbf{F}(z + y) + \int_z^\infty du \tilde{\mathbf{K}}(z, u) \mathbf{F}(u + y) = \mathbf{0}_{2 \times 2}$$

where $y > z$.

- 4 The solution $\mathbf{m}(z, t)$ of the HF is given by:

$$\mathbf{m}(z) \cdot \boldsymbol{\sigma} = \mathbf{H}(z) \sigma_3 \mathbf{H}(z)^{-1} = \left[I_2 + \tilde{\mathbf{K}}_{\bullet}(z) \right]^{-1} \sigma_3 \left[I_2 + \tilde{\mathbf{K}}_{\bullet}(z) \right].$$

Triplet Method

The **soliton solutions** are characterized by the condition $R(\lambda) = 0$.

In this case

$$F(w) = \sum_{s=1}^N e^{-\lambda_s w} c_s = C e^{-sA} B,$$

where A is a square matrix, B is a column vector and C is a row vector. **All the eigenvalues of A have positive real parts.**

The Marchenko's kernel can be written as:

$$F(w) = \begin{pmatrix} 0 & F(w) \\ -F(w)^* & 0 \end{pmatrix} = C e^{-wA} B,$$

where $\mathcal{A} = \begin{pmatrix} A & 0_{n \times n} \\ 0_{n \times n} & A^\dagger \end{pmatrix}$, $B = \begin{pmatrix} 0_{n \times 1} & B \\ -C^\dagger & 0_{n \times 1} \end{pmatrix}$, $C = \begin{pmatrix} C & 0_{1 \times n} \\ 0_{1 \times n} & B^\dagger \end{pmatrix}$.

For such kernels the Marchenko equation can be solved explicitly by separation of variables.

Explicit soliton solutions

Looking for a solution in the form

$$\tilde{\mathbf{K}}(z, y) = -\mathbf{L}(z)e^{-y\mathcal{A}}\mathcal{B},$$

we get

$$\tilde{\mathbf{K}}_{\bullet}(z) = -\mathcal{C}e^{-z\mathcal{A}} \left[I_{2n} + e^{-z\mathcal{A}}\mathcal{P}e^{-z\mathcal{A}} \right]^{-1} e^{-y\mathcal{A}}\mathcal{A}^{-1}\mathcal{B},$$

where

$$\mathcal{P} = \begin{pmatrix} 0_{q \times q} & N \\ -Q & 0_{q \times q} \end{pmatrix}, \quad N = \int_0^{\infty} dz e^{-z\mathcal{A}}\mathcal{B}\mathcal{B}^{\dagger}e^{-z\mathcal{A}^{\dagger}}, \quad Q = \int_0^{\infty} dz e^{-z\mathcal{A}^{\dagger}}\mathcal{C}^{\dagger}\mathcal{C}e^{-z\mathcal{A}}.$$

Time evolution:

$$(A, B, C) \mapsto (A, B, C e^{-4itA^2}).$$



HF solution:





$$\mathbf{m}(z, t) \cdot \boldsymbol{\sigma} = \left[I_2 + \tilde{\mathbf{K}}_{\bullet}(z, t) \right]^{-1} \sigma_3 \left[I_2 + \tilde{\mathbf{K}}_{\bullet}(z, t) \right].$$

We plan to extend, by following the same methodology, this study to the more general **Landau-Lifshitz** equation

$$m_t = m \wedge m_{zz} + m \wedge Jm$$

where $m(z, t) \in \mathbb{R}^3$ is a vector function satisfying some suitable asymptotic conditions and $J = \text{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix with $J_1 \leq J_2 \leq J_3$ and $J_1 < J_3$.

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Acknowledgments

Thank you!!