The inverse scattering transform for the defocusing nonlinear Schrödinger equation with nonzero boundary conditions

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(based on a joint work with B. Prinari, C. van der Mee, and F. Vitale)

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Introduction

In this talk we apply the Inverse Scattering Transform (IST) to solve the initial value problem of the defocusing NonLinear Schrödinger (NLS) equation with **N**onzero **B**oundary **C**onditions (**NZBCs**), i.e.

$$iq_t = q_{xx} - 2|q|^2 q$$

with NZBCs

$$q(x,t) \to q_\pm(t) = q_0 e^{2iq_0^2 t + i\theta_\pm} \quad \text{as } x \to \pm\infty,$$

where $q_0 > 0$ and $0 \leq \theta_\pm < 2\pi$ are arbitrary constants.

Introduction

This equation is important in many contexts related to nonlinear phenomena, such as:

- deep water waves;
- plasma physics;
- Bose-Einstein condensates;
- nonlinear fiber optics.

The interest in NLS as a prototypical integrable system is motivated because most dispersive energy preserving systems give rise, in appropriate limits, to the scalar NLS.

Introduction

Important dates about the application of the IST to defocusing NLS equations with NZBCs:

- 1973: Zakharov
- 1977-1978: Kawata and Inoue
- 1978-1983: Gerdjikov and Kulish
- 1980-1984: Leon, Asano and Kato, Boiti and Pempinelli
- 2006: Ablowitz, Biondini and Prinari
- 2011: Biondini, Prinari and Trubatch

Introduction: the Eigenvalue Problem

In order to solve the initial-value problem for the defocusing NLS equation by using the IST method, it is necessary to build the direct and inverse scattering for the following system (AKNS or ZS System):

$$\frac{\partial X}{\partial x}(x,k) = \left(-ik\sigma_3 + Q(x)\right) X(x,k)\,, \qquad x \in \mathbb{R},$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} 0 & q(x) \\ q^*(x) & 0 \end{pmatrix},$$

$q(x)$ is the potential (the our NZBCs), $q(x) - q_\pm$ belongs to $L^1(\mathbb{R}^\pm)$, k is a complex spectral parameter

For later convenience we write the AKNS system in the following equivalent form

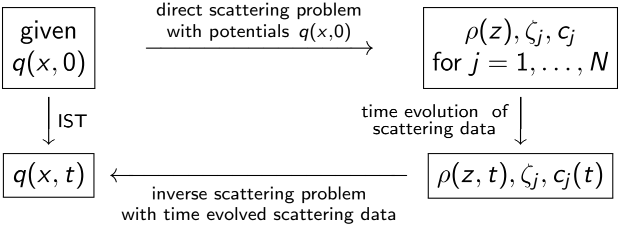
$$\frac{\partial X}{\partial x}(x,k) = A(x,k)X(x,k) + (Q(x) - Q_f(x))X(x,k)\,,$$

where we have defined

$$A(x,k) = \theta(x)A_+(k) + \theta(-x)A_-(k)\,, \qquad Q_f(x) = \theta(x)Q_+ + \theta(-x)Q_-\,,$$

$$A_\pm(k) = -ik\sigma_3 + Q_\pm \equiv \begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix}, \qquad Q_\pm = \begin{pmatrix} 0 & q_\pm \\ q_\pm^* & 0 \end{pmatrix}.$$

Introduction: Direct and Inverse Scattering of the AKNS System



THE DIRECT SCATTERING consists of: Determine the reflection coefficients, the bound states [poles of $\mathbf{t}_r(\lambda)$], and norming constants from the potentials $q(x)$.

THE INVERSE SCATTERING consists of: Reconstruct the potentials $q(x)$ from one reflection coefficient, the bound states, and the norming constants.

Introduction: Open problems and our contributions

Solving the defocusing NLS equation with NZBCs by the IST left many open problems so far. For example:

- No attempt has been made to identify the most suitable functional class of non-decaying potentials where the direct and inverse scattering problems can be solved;
- The analyticity properties of eigenfunctions and scattering data are not rigorously established;
- The possibility to have "purely radiative" solutions, i.e., solutions deriving only from the reflection coefficient without any contributions from the bound states, is not studied yet.

We will address all those problem and indicate some improvements. In particular, we will establish that the direct problem is well defined when $q - q_{\pm} \in L^{1,2}(\mathbb{R}^{\pm})$ and derive the analyticity properties of eigenfunctions and scattering data for potentials in this class in a rigorous way.

The spectral parameters k, λ

When we look for asymptotic eigenvalues and eigenvectors of the scattering problem, we have to deal with the new spectral variable $\lambda = \sqrt{k^2 - q_0^2}$.

The variable k is then thought of as belonging to a Riemann surface $\mathbb{K}$ consisting of a sheet $\mathbb{K}^+$ and a sheet $\mathbb{K}^-$ which both coincide with the complex plane cut along the semilines

$$\Sigma = (-\infty, -q_0] \cup [q_0, \infty)$$

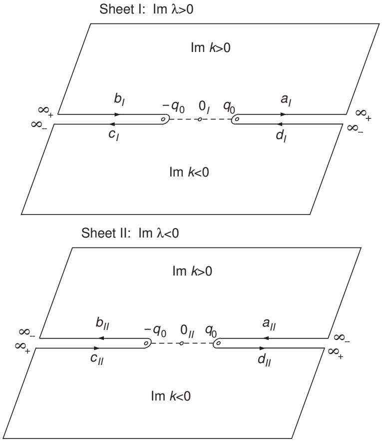
with its edges glued in such a way that $\lambda(k)$ is continuous through the cut.

The variable λ is thought of as belonging to the complex plane consisting of the upper half complex plane Λ^+ and the lower half complex plane Λ^- glued together along the full real line.

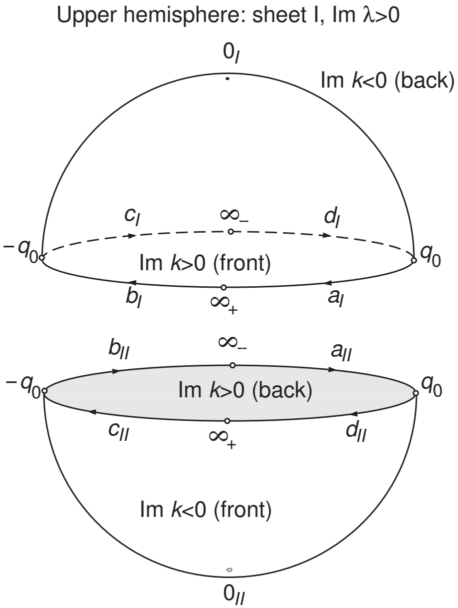
The spectral parameters k, λ

$$\lambda = \sqrt{k^2 - q_0^2}$$

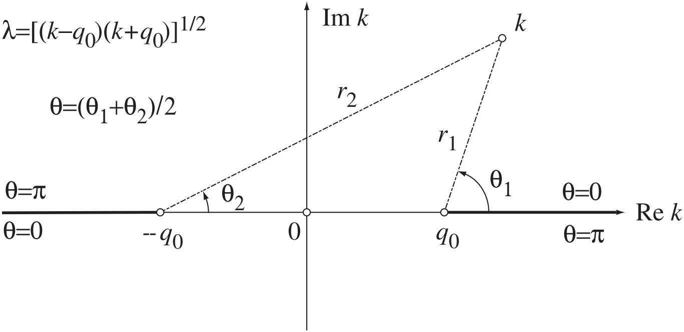
Many thanks to Barbara Prinari who gave me the following pictures.



The spectral parameters k, λ



The spectral parameters k, λ



Direct Problem: fundamental eigenfunctions

Let us consider the AKNS system

$$\frac{\partial X}{\partial x}(x,k) = A(x,k)X(x,k) + (Q(x) - Q_f(x))X(x,k)\,.$$

We define, for $k \in \Sigma$, the so-called *fundamental eigenfunctions* (from the right and from the left, respectively), in the following way

$$\tilde{\Psi}(x,k) = e^{xA_+(k)}[I_2 + o(1)], \qquad x \to +\infty,$$
$$\tilde{\Phi}(x,k) = e^{xA_-(k)}[I_2 + o(1)], \qquad x \to -\infty,$$

where I_p denotes the identity matrix of order p and

$$A_\pm(k) = \begin{pmatrix} -ik & q_\pm \\ q_\pm^* & ik \end{pmatrix}.$$

We want to know also the asymptotic behaviour of $\tilde{\Psi}(x,k)$ as $x \to -\infty$ and of $\tilde{\Phi}(x,k)$ for $x \to +\infty$.

Direct Problem: fundamental eigenfunctions

If the entries of $Q(x) - Q_f(x)$ *are in* $L^{1,2}(\mathbb{R})$, *for* $k \in \Sigma$, *the Volterra integral equations*

$$\tilde{\Psi}(x,k) = \mathcal{G}(x,0;k) - \int_x^\infty dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\tilde{\Psi}(y,k)\,,$$

$$\tilde{\Phi}(x,k) = \mathcal{G}(x,0;k) + \int_{-\infty}^x dy\, \mathcal{G}(x,y;k)[Q(y) - Q_f(y)]\tilde{\Phi}(y,k)\,,$$

have the fundamental eigenfunctions before defined as their unique solutions.

Now it is easy to get the asymptotic behaviour of $\Psi(x,k)$ and $\Phi(x,k)$.

$$\tilde{\Psi}(x,k) = \mathcal{G}(x,0;k)[\mathbb{A}_l(k) + o(1)], \qquad x \to -\infty,$$

$$\tilde{\Phi}(x,k) = \mathcal{G}(x,0;k)[\mathbb{A}_r(k) + o(1)], \qquad x \to +\infty,$$

where the *transition coefficient matrices* $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ are given by

$$\mathbb{A}_l(k) = I_2 - \int_{-\infty}^\infty dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde{\Psi}(y,k),$$

$$\mathbb{A}_r(k) = I_2 + \int_{-\infty}^\infty dy\, \mathcal{G}(0,y;k)[Q(y) - Q_f(y)]\tilde{\Phi}(y,k).$$

Fundamental Matrix

The matrix function $\mathcal{G}(x, y; k)$ is called *fundamental matrix* for the scattering problem with generator $A(x, k)$. It is a solution of the AKNS system with potential $Q(x) = Q_f(x)$ and satisfies

$$\frac{\partial}{\partial x}\mathcal{G}(x, y; k) = A(x, k)\mathcal{G}(x, y; k)\,,$$
$$\mathcal{G}(x, x; k) = I_2\,.$$

One has

$$\mathcal{G}(x, y; k) = \begin{cases} e^{(x-y)A_+(k)}, & x, y \geq 0, \\ e^{(x-y)A_-(k)}, & x, y \leq 0, \\ e^{xA_+(k)}e^{-yA_-(k)}, & x, -y \geq 0, \\ e^{xA_-(k)}e^{-yA_+(k)}, & x, -y \leq 0. \end{cases}$$

Note that $\mathcal{G}(x, y; k)$ is a square matrix which depends continuously on $(x, y, k) \in \mathbb{R}^2 \times \Sigma$. An important property is the following

$$\|\mathcal{G}(x, y; k)\| \leq \begin{cases} C, & k < -q_0 \text{ or } k > q_0, \\ C(1 + |x|)(1 + |y|), & k = \pm q_0. \end{cases}$$

where $C \geq 1$ is a constant (independent of $(x, y) \in \mathbb{R}^2$).

Direct Problem: Jost solutions

Let us consider the following matrix

$$W_{\pm}(k) = \begin{pmatrix} \lambda + k & \lambda - k \\ iq_{\pm}^{*} & -iq_{\pm}^{*} \end{pmatrix} ,$$

with $\det W_{\pm}(k) = -2iq_{\pm}^{*}\lambda$ and $A_{\pm}(k)W_{\pm}(k) = W_{\pm}(k)\mathrm{diag}(-i\lambda, i\lambda)$.
We introduce the *Jost solutions* from the right and the left, respectively, as

$$\tilde{\Psi}(x,k)W_{+}(k) = \begin{pmatrix} \bar{\psi}(x,k) & \psi(x,k) \end{pmatrix} ,$$
$$\tilde{\Phi}(x,k)W_{-}(k) = \begin{pmatrix} \phi(x,k) & \bar{\phi}(x,k) \end{pmatrix} .$$

We get for the Jost solutions

$$\bar{\psi}(x,k) \sim e^{-i\lambda x} \begin{pmatrix} \lambda + k \\ iq_{+}^{*} \end{pmatrix}, \qquad \psi(x,k) \sim e^{i\lambda x} \begin{pmatrix} \lambda - k \\ -iq_{+}^{*} \end{pmatrix}, \quad x \to +\infty,$$
$$\phi(x,k) \sim e^{-i\lambda x} \begin{pmatrix} \lambda + k \\ iq_{-}^{*} \end{pmatrix}, \qquad \bar{\phi}(x,k) \sim e^{i\lambda x} \begin{pmatrix} \lambda - k \\ -iq_{-}^{*} \end{pmatrix}, \quad x \to -\infty.$$

Direct Problem: Jost solutions

Since $\tilde{\Psi}(x,k)$ and $\tilde{\Phi}(x,k)$ are square matrix solutions of the AKNS system (which is a homogeneous first order system), we have

$$\tilde{\Psi}(x,k) = \tilde{\Phi}(x,k)\mathbb{A}_l(k), \qquad \tilde{\Phi}(x,k) = \tilde{\Psi}(x,k)\mathbb{A}_r(k),$$

where $\mathbb{A}_l(k)$ and $\mathbb{A}_r(k)$ are the transition coefficient matrices. We easily get

$$\begin{pmatrix}\phi(x,k) & \bar{\phi}(x,k)\end{pmatrix} = \begin{pmatrix}\bar{\psi}(x,k) & \psi(x,k)\end{pmatrix} S(k)\,,$$
$$\begin{pmatrix}\bar{\psi}(x,k) & \psi(x,k)\end{pmatrix} = \begin{pmatrix}\phi(x,k) & \bar{\phi}(x,k)\end{pmatrix} \bar{S}(k)\,,$$

where

$$S(k) = W_+^{-1}(k)\mathbb{A}_r(k)W_-(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k)\end{pmatrix}, \quad \bar{S}(k) = S^{-1}(k).$$

RED and BLU stress the different property of analiticity of the Jost solutions. Under the hypothesys that $Q(x) - Q_f(x)$ belongs to $L^{1,2}(\mathbb{R})$, RED denotes continuity for $k \in \overline{\mathbb{K}^+}$ and analiticity for $k \in \mathbb{K}^+$, while BLU continuity for $k \in \overline{\mathbb{K}^-}$ and analytic for $k \in \mathbb{K}^-$.

Direct problem: Scattering matrix

Taking into account the analyticity properties of the Jost solutions, it is convenient to consider the following matrix functions

$$\begin{pmatrix}\phi(x,k) & \psi(x,k)\end{pmatrix}, \quad \begin{pmatrix}\bar{\psi}(x,k) & \bar{\phi}(x,k)\end{pmatrix}.$$

In fact they allow us to formulate the Riemann-Hilbert problems

$$\begin{aligned}\begin{pmatrix}\phi(x,k) & \psi(x,k)\end{pmatrix} &= \begin{pmatrix}\bar{\psi}(x,k) & \bar{\phi}(x,k)\end{pmatrix}\sigma_3\bar{\mathbf{T}}(k)\sigma_3,\\ \begin{pmatrix}\bar{\psi}(x,k) & \bar{\phi}(x,k)\end{pmatrix} &= \begin{pmatrix}\phi(x,k) & \psi(x,k)\end{pmatrix}\sigma_3\mathbf{T}(k)\sigma_3,\end{aligned}$$

where $\mathbf{T}(k) = \begin{pmatrix} t_l(k) & r(k) \\ \rho(k) & t_r(k)\end{pmatrix}$ and $\bar{\mathbf{T}}(k) = \begin{pmatrix} \bar{t}_r(k) & \bar{\rho}(k) \\ \bar{r}(k) & \bar{t}_l(k)\end{pmatrix}$.

The scattering data consists of: one of the **reflection coefficient**, **the bound states** ζ_j, i.e, the poles of the transmission coefficient $t_l(k)$ (or $t_r(k)$) and a suitable set of constants c_j associated to the bound states, the so-called **norming constants**.

Properties of the bound states

It is already known in literature that the bound states are **simple**. But we have also the following results:

- If $Q(x) - Q_f(x)$ belongs to $L^{1,4}(\mathbb{R})$, then the bound states are finite in number, all of them belonging to spectral gap $k \in (-q_0, q_0)$.
- Let $0 < \gamma^2 < 1$ be a constant. If

$$\int_{-\infty}^{0} dx\, |q(x) - q_-| + \int_{0}^{\infty} dx\, |q(x) - q_+| < \gamma^2 \frac{\pi}{2}\,,$$

  there do not exist any discrete eigenvalues for

$$k \in \left(-q_0\sqrt{1-\gamma^2}, q_0\sqrt{1-\gamma^2}\right)\,.$$

The variable z

To formulate and solve the inverse problem, it is more convenient to use uniformization variable z defined by the conformal mapping:

$$z = k + \lambda(k)\,,$$

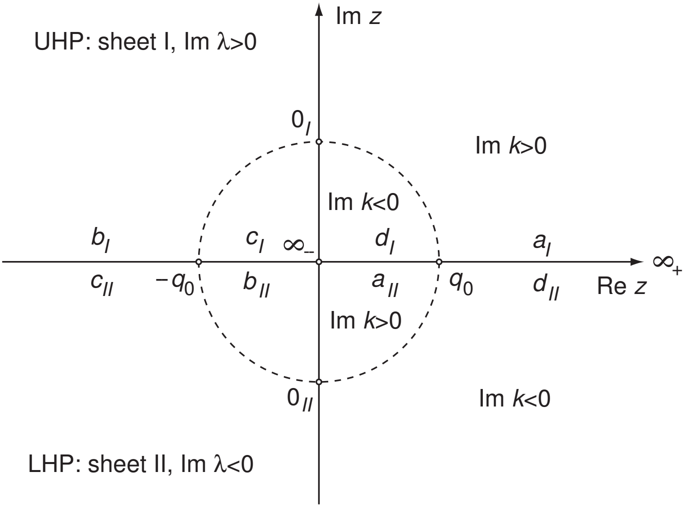
and inverse mapping given by

$$k = \frac{1}{2}\left(z + \frac{q_0^2}{z}\right)\,, \qquad \lambda = z - k = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right)\,.$$

We observe that

- the two sheets $\mathbb{K}^+$, $\mathbb{K}^-$ of the Riemann surface $\mathbb{K}$ are, respectively, mapped onto the upper and lower half-planes $\mathbb{C}^\pm$ of the complex z-plane;
- the cut Σ on the Riemann surface is mapped onto the real z axis;
- the segments $-q_0 \leq k \leq q_0$ on $\mathbb{K}^+$ and $\mathbb{K}^-$ are mapped onto the upper and lower semicircles of radius q_0 and center at the origin of the z-plane.

The variable z



Inverse Problem

In order to (re)-construct the potential we use the well-known method based on the solution of the so-called Marchenko integral equation.

1. Given the scattering data $\left\{\rho(z)\,,\{\zeta_j\}_{j=1}^{N},\,\{c_j\}_{j=1}^{N}\right\}$, we build the kernel $\mathbb{G}(x+y)$.

Inverse Problem

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1. Given the scattering data $\left\{\rho(z)\,,\{\zeta_j\}_{j=1}^N,\ \{c_j\}_{j=1}^N\right\}$, we build the kernel $\mathbb{G}(x+y)$.

2. Using the matrix function $\mathbb{G}(x+y)$, we can consider the following Marchenko equation

$$\mathbf{K}(x,y)+\mathbb{G}(x+y)+\int_x^\infty ds\,\mathbf{K}(x,s)\mathbb{G}(s+y)=0\,.$$

Inverse Problem

In order to (re)-construct the potential we use the well-known method based on the solution of the so-called Marchenko integral equation.

1. Given the scattering data $\left\{\rho(z), \{\zeta_j\}_{j=1}^N, \{c_j\}_{j=1}^N\right\}$, we build the kernel $\mathbb{G}(x+y)$.

2. Using the matrix function $\mathbb{G}(x+y)$, we can consider the following Marchenko equation

$$\mathbf{K}(x,y) + \mathbb{G}(x+y) + \int_x^\infty ds\, \mathbf{K}(x,s)\mathbb{G}(s+y) = 0 \, .$$

3. The potentials $u(x)$ is connected to the above equations by means of the following relationship (which will appear more clear later)

$$q(x) = q_+ - 2K_{12}(x,x) \, .$$

Inverse Problem

The Marchenko equation for the NLS equations with NZBCs are

$$\mathbf{K}(x,y) + \mathbb{G}(x+y) + \int_x^\infty ds\, \mathbf{K}(x,s)\mathbb{G}(s+y) = 0\,,$$

where $\mathbf{K}(x,y)$ and $\mathbb{G}(s+y)$ are defined as

$$\mathbf{K}(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix}\,,\ \mathbb{G}(s+y) = \begin{pmatrix} F_1(s+y) & F_2^*(s+y) \\ F_2(s+y) & F_1^*(s+y) \end{pmatrix}$$

$$F_1(x) = F_{1,c}(x) + iF'_{2,c}(x) - \tfrac{\zeta_n^*}{2} F_{1,d}(x)\,,$$
$$F_2(x) = -iq_+^*\left[F_{2,c}(x) + \tfrac{1}{2} F_{1,d}(x)\right]\,,$$

where

$$F_{1,c}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\, e^{i\zeta x}\, \frac{\rho(\sqrt{\zeta^2+q_0^2},\zeta) + \rho(-\sqrt{\zeta^2+q_0^2},\zeta)}{2}\,,$$

$$F_{2,c}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\, e^{i\zeta x}\, \frac{\rho(\sqrt{\zeta^2+q_0^2},\,\zeta) - \rho(-\sqrt{\zeta^2+q_0^2},\zeta)}{2\sqrt{\zeta^2+q_0^2}}\,,$$

$$F_{1,d}(x) = -i\sum_{n=1}^{N} c_n\, e^{-\nu_n x}\,,\ \ \zeta_n = k_n + i\nu_n \,\text{discrete eigenvalues.}$$

Inverse problem: the triplet method

Now we want to solve explicitly the Marchenko equation in the reflectionless case (multisoliton solutions).

We use the triplet method already used to solve, for example, the NLS equation under vanishing boundary conditions and the sine-Gordon equation.

The main advantages of this method are:

1. It is applicable to other integrable nonlinear evolution equations (KdV, mKdV, sine-Gordon).
2. The explicit formula found is expressed in a concise form in terms of the triplet (A, B, C). Using computer algebra, we can "unzip" the solution in terms of exponential, trigonometric, and polynomial functions of x and t. Even for matrices A of moderate order, this unzipped expression may take several pages!
3. Choosing different triplets as input in our formula, we get a set of solutions to the NLS equation which can be used for "validation" of numerical methods.

Inverse problem

In the (reflectionless case)

$$\mathbb{G}(z) = \mathbf{C}e^{-z\mathbf{A}}\mathbf{B}\,,$$

$\mathbf{A}$ is a $p \times p$ matrix having only eigenvalues with positive real part, $\mathbf{B}$ is a $p \times 2$ matrix, and $\mathbf{C}$ is a $2 \times p$ matrix.

Let us also assume that all the eigenvalues of $\mathbf{A}$ have positive real parts and $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a minimal triplet, i.e.,

$$\bigcap_{r=1}^{+\infty} \ker \mathbf{C}\mathbf{A}^{r-1} = \bigcap_{r=1}^{+\infty} \ker \mathbf{B}^{\dagger}(\mathbf{A}^{\dagger})^{r-1} = \{0\}.$$

It is noteworthy that the triplet yielding a minimal realization is unique up to a similarity transformation $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \to (\mathbf{S}\mathbf{A}\mathbf{S}^{-1}, \mathbf{S}\mathbf{B}, \mathbf{C}\mathbf{S}^{-1})$ for some unique matrix $\mathbf{S}$.

Inverse problem

Putting

$$\mathbb{G}(z) = \mathbf{C}e^{-z\mathbf{A}}\mathbf{B}$$

into the Marchenko equation we obtain

$$\mathbf{K}(x,y) = -\left[\mathbf{C}e^{-x\mathbf{A}} + \int_x^\infty ds\,\mathbf{K}(x,s)\mathbf{C}e^{-s\mathbf{A}}\right]e^{-y\mathbf{A}}\mathbf{B} = -\left[\mathbf{C}e^{-x\mathbf{A}} + \mathbf{L}(x)\right]e^{-y\mathbf{A}}\mathbf{B},$$

where

$$\mathbf{L}(x) = \int_x^\infty ds\,\mathbf{K}(x,s)\mathbf{C}e^{-s\mathbf{A}}.$$

Defining

$$\mathbf{P} = \int_0^\infty dz\,e^{-z\mathbf{A}}\mathbf{B}\mathbf{C}e^{-z\mathbf{A}},$$

we arrive, after some easy and straightforward calculations, at the following expression for

$$\mathbf{L}(x) = -\mathbf{C}e^{-2x\mathbf{A}}\mathbf{P}e^{-x\mathbf{A}}[I_p + e^{-x\mathbf{A}}\mathbf{P}e^{-x\mathbf{A}}]^{-1},$$

and, consequently

$$\mathbf{K}(x,y) = -\mathbf{C}e^{-x\mathbf{A}}[I_p + e^{-x\mathbf{A}}\mathbf{P}e^{-x\mathbf{A}}]^{-1}e^{-y\mathbf{A}}\mathbf{B}.$$

Inverse Problem

Writing

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}^{(1)} \\ \mathbf{C}^{(2)} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{B}^{(1)} & \mathbf{B}^{(2)} \end{pmatrix},$$

where $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$ are rows of length p and $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ are columns of length p, we get

$$\begin{aligned} q(x) &= q_+ + 2\mathbf{C}^{(1)} e^{-x\mathbf{A}} [I_p + e^{-x\mathbf{A}} \mathbf{P} e^{-x\mathbf{A}}]^{-1} e^{-x\mathbf{A}} \mathbf{B}^{(2)} \\ &= q_+ + 2\mathbf{C}^{(1)} [\mathbf{P} + e^{2x\mathbf{A}}]^{-1} \mathbf{B}^{(2)} \end{aligned}$$

We observe that the above equation yields

$$q_- = q_+ + 2\mathbf{C}^{(1)} \mathbf{P}^{-1} \mathbf{B}^{(2)}$$

in the limit $x \to -\infty$ which requires knowing that $\mathbf{P}$ is invertible.

Inverse Problem

Note that for fixed $x \in \mathbb{R}$, the existence of the inverse $e^{2x\mathbf{A}} + \mathbf{P}$ is equivalent to the unique solvability of the Marchenko equation.

In order to have solutions of the NLS with nonvanishing boundary conditions, we have to assume

1. the minimality of the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$;
2. the positivity of the real parts of the eigenvalues of the matrix $\mathbf{A}$;
3. the invertibility of the matrices $e^{2x\mathbf{A}} + \mathbf{P}$ and $\mathbf{P}$.

If $\mathbf{P}$ is an invertible matrix, then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a minimal triplet. The viceversa is not true.

Inverse Problem: evolution of the scattering data

The evolution of the scattering is well-known in literature. In particular,

- the discrete eigenvalues $-q_0 < k_n < q_0$ are time-independent,
- the time dependence of the reflection coefficients satisfy $\rho(t) = \rho(0)e^{-4ik\lambda t}$
- the norming constants evolve as $C_n(t) = C_n(0)e^{4k_n\nu_n t}$.

In the reflectionless case, we can write the elements of the matrix $\mathbb{G}(x,t)$ as

$$F_1(x,t) = \frac{i}{2}\sum_{n=1}^{N} C_n(t)\zeta_n^* e^{-\nu_n x}\,, \qquad F_2(x,t) = -\frac{q_+^*}{2}\sum_{n=1}^{N} C_n(t)e^{-\nu_n x}\,.$$

Inverse Problem: evolution of the scattering data

We have

$$\mathbb{G}(x,t) = \frac{1}{2}\sum_{n=1}^{N} e^{-\nu_n x} \begin{pmatrix} iC_n(t)\zeta_n^* & -q_+(t)C_n^*(t) \\ -q_+^*(t)C_n(t) & -i\zeta_n C_n^*(t) \end{pmatrix} = \mathbf{C}(t)e^{-x\mathbf{A}}\mathbf{B}(t)\,,$$

where $\mathbf{A} = \operatorname{diag}(\nu_1, \ldots, \nu_N)$,

$$\mathbf{B}(t) = \frac{1}{2}\begin{pmatrix} i\zeta_1^* C_1(t) & -q_+(t)C_1^*(t) \\ \vdots & \vdots \\ i\zeta_N^* C_N(t) & -q_+(t)C_N^*(t) \end{pmatrix}\,,$$

$$\mathbf{C}(t) = \begin{pmatrix} 1 & \ldots & 1 \\ \dfrac{i\zeta_1}{q_+(t)} & \ldots & \dfrac{i\zeta_N}{q_+(t)} \end{pmatrix}\,.$$

Then

$$\mathbf{P}(t) = \int_0^\infty dx\, e^{-x\mathbf{A}}\mathbf{B}(t)\mathbf{C}(t)e^{-x\mathbf{A}}\,.$$

Inverse Problem

To write down the solution $q(x,t)$ of the NLS equation with NZBCs boundary conditions at the generic time t (in the reflectionless case), it suffices to use the triplet $(\mathbf{A},\mathbf{B}(t),\mathbf{C}(t))$ and the matrix $\mathbf{P}(t)$, instead of $(\mathbf{A},\mathbf{B},\mathbf{C})$ and $\mathbf{P}$ in the expression of $q(x)$, obtaining

$$
\begin{aligned}
q(x;t) &= q_+ + 2\mathbf{C}(t)^{(1)} e^{-x\mathbf{A}}[I_p + e^{-x\mathbf{A}}\mathbf{P}(t)e^{-x\mathbf{A}}]^{-1} e^{-x\mathbf{A}}\mathbf{B}(t)^{(2)} \\
&= q_+ + 2\mathbf{C}(t)^{(1)}[\mathbf{P}(t) + e^{2x\mathbf{A}}]^{-1}\mathbf{B}(t)^{(2)}
\end{aligned}
$$

Inverse Problem: One example

We want to find the one soliton solution by using the triplet method.

Choosing the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ as:

$$\mathbf{A} = (\nu_1), \quad \mathbf{B} = \frac{1}{2}\begin{pmatrix} ic_1\zeta_1^* & -q_+ c_1^* \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 \\ \frac{i\zeta_1}{q_+} \end{pmatrix}.$$

As a result, $\mathbf{P} = (ic_1\zeta_1^* - ic_1^*\zeta_1)/(4\nu_1)$. The one soliton solution is given by:

$$q(x,t) = q_+(t)\left[1 - \frac{\bar{C}_1(0)}{\zeta_1}\frac{e^{-2\nu_1 x + 4k_1\nu_1 t}}{1 + \frac{\bar{C}_1(0)}{2\nu_1}e^{-2\nu_1 x + 4k_1\nu_1 t}}\right].$$

Note this solution coincides with the solution obtained by solving the RH problem.

Thank you for your attention!!!