Direct Scattering Problem for AKNS system: characterization of scattering data

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Contents

Introduction: AKNS system

In this talk we study the so-called "characterization problem" for the AKNS system, i.e. the following system of linear differential equation (LODE):

$$iJ\frac{\partial X}{\partial x}(\lambda, x) - V(x)X(\lambda, x) = \lambda X(\lambda, x),$$

where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0_{m\times m} & iq(x) \\ ir(x) & 0_{n\times n} \end{pmatrix},$$

the potentials $q(x)$ and $r(x)$ have their entries in $L^1(\mathbb{R})$, and λ is a spectral parameter.

- For $n = m = 1$ this system is called the Zakharov-Shabat system;
- For $n = 1$ and $m = 2$ we have the Manakov system.

In the *defocusing case* we have $r(x) = -q(x)^\dagger$ and hence $V(x)^\dagger = V(x)$; in the *focusing case* $r(x) = q(x)^\dagger$ and hence $V(x)^\dagger = -V(x)$.

Introduction: IST

The initial value problem of many nonlinear evolution equation can be solved by the Inverse Scattering Transform (IST).

1. Let us consider the initial value problem for the matrix NLS system

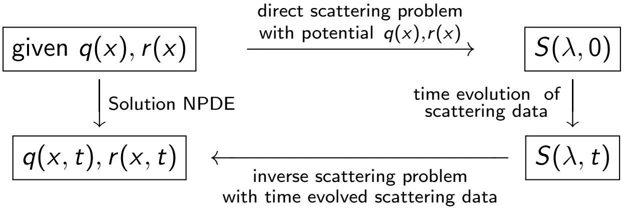
$$\begin{cases} i\,u_t + u_{xx} - 2uvu = 0_{m\times n}, \\ -i\,v_t + v_{xx} - 2vuv = 0_{n\times m}, \\ u(x,0) = q(x), \qquad v(x,0) = r(x). \end{cases}$$

2. We associate to this equation the AKNS system

$$iJ\frac{\partial X}{\partial x}(\lambda, x) - V(x)X(\lambda, x) = \lambda X(\lambda, x).$$

Remark: The AKNS system appear as the LODE associated to other evolution equation such as the modified Korteweg-de Vries equation, the sine-Gordon and Hirota equation.

Introduction: IST



Direct Scattering Problem: Given the potentials $\{q(x), r(x)\}$ construct the scattering data $\left\{R(\lambda), \{\lambda_j, c_j\}_{j=1}^{N}\right\}$

Inverse Scattering Problem: Given the (evolved) scattering data $\left\{R(\lambda,t), \{\lambda_j, c_j(t)\}_{j=1}^{N}\right\}$ construct the potentials $\{q(x,t), r(x,t)\}$.

Introduction: What's the problem

Characterization problem

The characterization problem can be described as follows: *Give necessary and sufficient conditions for a matrix*

$$\begin{pmatrix} a(k) & b(k) \\ c(k) & d(k) \end{pmatrix}$$

to be the scattering matrix of a potential $V(x) \in L^1(\mathbb{R})$.

$q(x),\ r(x) \in L^1$

direct scattering problem $\longrightarrow$

$\longleftarrow$ inverse scattering problem

$R(k)$ or $L(k)$
plus
bound state data

Introduction: History of the problem

An analogous characterization problem can be studied for the Schrödinger equation on the line. This characterization problem has been solved by:

- Marchenko (1986) for potentials in $L^1(\mathbb{R};(1+x^2)dx)$;
- Melin (1985) for potential in $L^1(\mathbb{R};(1+|x|)dx)$:

For the AKNS systems only partial results are known.

- Melik-Adamyan (1989) on the half-line (focusing case) ;
- Aktosun, Klaus, van der Mee (2000) (defocusing case);
- van der Mee (2004) (focusing case without bound states);
- Ablowitz, Prinari and Villarroel (2005) (focusing case);

REMARK: In all those papers the characterization results regard $1,1$-correspondences between a class of potentials and a class of scattering data *without involving the time variable in any respect*.

Introduction: Reformulation of the problem

We build a $1,1$-correspondences as denoted below

$$\boxed{q(x),\ \ r(x) \in L^1} \quad \underset{\text{inverse\ \ scattering problem}}{\overset{\text{direct scattering problem}}{\rightleftarrows}} \quad \boxed{\{\Omega(x+y), \breve{\Omega}(x+y)\}}$$

where $\{\Omega(x+y), \breve{\Omega}(x+y)\}$ are the kernel of the Marchenko integral equations and they enclose the scattering data:

$$\Omega(x+y) = \rho(x+y) + Ce^{-(x+y)A}B,$$
$$\breve{\Omega}(x+y) = \breve{\rho}(x+y) + \breve{C}e^{-(x+y)\breve{A}}\breve{B},$$

with $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ (they are the Fourier transforms of the reflection coefficients) and (A, B, C) and $(\breve{A}, \breve{B}, \breve{C})$ are triplets of size compatible matrices (A and $\breve{A}$ have only eigenvalues with positive real parts).

Direct Scattering Problem: Jost solutions

We define the *Jost matrices* $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ from the right and the left as those solutions to the AKNS system satisfying the asymptotic conditions

$$\Psi(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix} = e^{-i\lambda Jx}[I_{m+n} + o(1)], \qquad x \to +\infty,$$
$$\Phi(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix} = e^{-i\lambda Jx}[I_{m+n} + o(1)], \qquad x \to -\infty,$$

We get the following Volterra integral equations

$$\Psi(\lambda,x) = e^{-i\lambda Jx} + iJ\int_x^\infty dy\, e^{i\lambda J(y-x)} V(y)\Psi(\lambda,y),$$
$$\Phi(\lambda,x) = e^{-i\lambda Jx} - iJ\int_{-\infty}^x dy\, e^{-i\lambda J(x-y)} V(y)\Phi(\lambda,y).$$

Then

$$\Phi(\lambda,x) = \Psi(\lambda,x)a_r(\lambda), \qquad \Psi(\lambda,x) = \Phi(\lambda,x)a_l(\lambda).$$

$a_l(\lambda)$ and $a_r(\lambda) = a_l^{-1}(\lambda)$ are the *transition matrices* from the left and the right, respectively.

Direct Scattering Problem: analytic properties

$$\Psi(\lambda,x)=\begin{pmatrix}\overline{\psi}_1(\lambda,x) & \psi_2(\lambda,x)\\ \overline{\psi}_3(\lambda,x) & \psi_4(\lambda,x)\end{pmatrix},\ \Phi(\lambda,x)=\begin{pmatrix}\phi_1(\lambda,x) & \overline{\phi}_2(\lambda,x)\\ \phi_3(\lambda,x) & \overline{\phi}_4(\lambda,x)\end{pmatrix},$$

where the matrices in *RED* are continuous in $\lambda\in\overline{\mathbb{C}^-}$ and analytic in $\mathbb{C}^-$, while *BLU* matrices are continuous in $\lambda\in\overline{\mathbb{C}^+}$ and analytic in $\mathbb{C}^+$.
Similarly, for the transmission matrices:

$$a_l(\lambda)=\begin{pmatrix}a_{l1}(\lambda) & a_{l2}(\lambda)\\ a_{l3}(\lambda) & a_{l4}(\lambda)\end{pmatrix}\quad a_r(\lambda)=\begin{pmatrix}a_{r1}(\lambda) & a_{r2}(\lambda)\\ a_{r3}(\lambda) & a_{r4}(\lambda)\end{pmatrix}$$

From now on, we assume that there are no spectral singularities, i.e., the matrices $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$ are invertible matrices for all $\lambda\in\mathbb{R}$.
The points $\lambda\in\mathbb{C}^\pm$, where $a_{l1}(\lambda)$, $a_{r4}(\lambda)$, $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are singular matrices, are called isolated eigenvalues (bound states) of the AKNS system.

Direct Scattering Problem: the scattering matrices

Introducing

$$F_+(x,\lambda) = \begin{pmatrix} \overline{\psi}_1(\lambda,x) & \overline{\phi}_2(\lambda,x) \\ \overline{\psi}_3(\lambda,x) & \overline{\phi}_4(\lambda,x) \end{pmatrix}, \; F_-(x,\lambda) = \begin{pmatrix} \phi_1(\lambda,x) & \psi_2(\lambda,x) \\ \phi_3(\lambda,x) & \psi_4(\lambda,x) \end{pmatrix},$$

we arrive at the following Riemann-Hilbert problem:

$$F_-(\lambda,x) = F_+(\lambda,x) J S(\lambda) J, \qquad F_+(\lambda,x) = F_-(\lambda,x) J \check{S}(\lambda) J,$$

where the scattering matrices $S(\lambda) = \check{S}^{-1}(\lambda)$ are continuous functions of $\lambda \in \mathbb{R}$

$$S(\lambda) = \begin{pmatrix} T_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix}, \qquad \check{S}(\lambda) = \begin{pmatrix} \check{T}_l(\lambda) & \check{R}(\lambda) \\ \check{L}(\lambda) & \check{T}_r(\lambda) \end{pmatrix},$$

where

$$\begin{aligned} T_r(\lambda) &= a_{r1}(\lambda)^{-1}, \\ T_l(\lambda) &= a_{l4}(\lambda)^{-1}, \\ R(\lambda) &= -a_{l4}(\lambda)^{-1} a_{l3}(\lambda) = a_{r3}(\lambda) a_{r1}(\lambda)^{-1}, \\ L(\lambda) &= -a_{r1}(\lambda)^{-1} a_{r2}(\lambda) = a_{l2}(\lambda) a_{l4}(\lambda)^{-1}. \end{aligned}$$

Jost solutions and potentials

$$\Psi(\lambda, x) = e^{-i\lambda Jx} + \int_x^\infty dy \, \left(\overline{K}(x,y) \quad K(x,y)\right) e^{-i\lambda Jy},$$

$$\Phi(\lambda, x) = e^{-i\lambda Jx} + \int_{-\infty}^x dy \, \left(M(x,y) \quad \overline{M}(x,y)\right) e^{-i\lambda Jy},$$

where $\overline{K}(x,y) = \begin{pmatrix} \overline{K}^{up}(x,y) \\ \overline{K}^{dn}(x,y) \end{pmatrix}$ and $M(x,y) = \begin{pmatrix} M^{up}(x,y) \\ M^{dn}(x,y) \end{pmatrix}$ are $(m+n) \times m$ matrices, while $K(x,y) = \begin{pmatrix} K^{up}(x,y) \\ K^{dn}(x,y) \end{pmatrix}$ and $\overline{M}(x,y) = \begin{pmatrix} \overline{M}^{up}(x,y) \\ \overline{M}^{dn}(x,y) \end{pmatrix}$ are $(m+n) \times n$ matrices (*up* denotes the first m rows of a given matrix).

The potential pair $\{q(x), r(x)\}$ is related to the matrix functions $\overline{K}(x,y), \overline{M}(x,y), K(x,y), M(x,y)$ in the following way:

$$q(x) = -2K^{up}(x,x) \;\; = 2\overline{M}^{up}(x,x)$$

$$r(x) = 2\overline{K}^{dn}(x,x) \;\; = -2M^{dn}(x,x)$$

Marchenko equations

For $y \geq x$, we get the following Marchenko equations

$$\overline{K}(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \Omega(x+y) + \int_x^\infty dz\, K(x,z)\Omega(z+y) = 0_{(m+n)\times m},$$

$$K(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \breve{\Omega}(x+y) + \int_x^\infty dz\, \overline{K}(x,z)\breve{\Omega}(z+y) = 0_{(m+n)\times n},$$

where $\Omega(x), \breve{\Omega}(x)$ are, respectively, $n \times m$ and $m \times n$ matrix functions, and if all poles are simple

$$\Omega(x+y) = \rho(x+y) + \sum_{j=1}^{N} \mathbf{C}_j e^{-\kappa_j x}, \quad \breve{\Omega}(x+y) = \breve{\rho}(x+y) + \sum_{j=1}^{N} \breve{\mathbf{C}}_j e^{-\breve{\kappa}_j x},$$

while, if the poles are no simple

$$\Omega(x+y) = \rho(x+y) + \sum_{j=1}^{N} \sum_{l=0}^{\nu_j - 1} \mathbf{C}_{j,l} \frac{x^l}{l!} e^{-\kappa_j x},$$

$$\breve{\Omega}(x+y) = \breve{\rho}(x+y) + \sum_{j=1}^{N} \sum_{l=0}^{\nu_j - 1} \breve{\mathbf{C}}_{j,l} \frac{x^l}{l!} e^{-\breve{\kappa}_j x}.$$

$\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$.

Marchenko equations

IDEA: Represent the discrete part of the kernels in a more suitable way by using the exponential matrix!.

Let (A, B, C) and $(\breve{A}, \breve{B}, \breve{C})$ be triplets of size compatible matrices such that A and $\breve{A}$ only have eigenvalues with positive real parts.

By expanding the matrix exponentials for matrices A, the bound state terms can be written in the form

$$\sum_{j=1}^{N} \sum_{l=0}^{\nu_j - 1} \mathbf{C}_{j,l} \frac{s^l}{l!} e^{-\kappa_j s} = C e^{-sA} B,$$

where $\kappa_1, \ldots, \kappa_N$ are distinct numbers with positive real parts, ν_j are the orders of the poles of the transmission coefficient at the discrete eigenvalues $i\kappa_j$, and $\mathbf{C}_{j,l}$ are the so-called norming constants.

We consider the kernels in the form

$$\Omega(x+y) = \rho(x+y) + C e^{-(x+y)A} B,$$

$$\breve{\Omega}(x+y) = \breve{\rho}(x+y) + \breve{C} e^{-(x+y)\breve{A}} \breve{B},$$

OBSERVATION: The kernels above introduced contain all the scattering data and are suitable to solve the characterization problem.

Marchenko equations

For $y \leq x$, we have another pair of Marchenko equations:

$$M(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \breve{\Xi}(x+y) + \int_{-\infty}^{x} dz\, \overline{M}(x,z)\breve{\Xi}(x+y) = 0_{(m+n)\times m},$$
$$\overline{M}(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \Xi(x+y) + \int_{-\infty}^{x} dz\, M(x,z)\Xi(z+y) = 0_{(m+n)\times n},$$

$$\Xi(x+y) = \ell(x+y) + \mathcal{C}e^{(x+y)\mathcal{A}}\mathcal{B},$$
$$\breve{\Xi}(x+y) = \breve{\ell}(x+y) + \breve{\mathcal{C}}e^{(x+y)\breve{\mathcal{A}}}\breve{\mathcal{B}},$$

$\breve{\Xi}(x), \Xi(x)$ are, respectively, $n \times m$ and $m \times n$ matrix functions and $\ell(x)$ and $\breve{\ell}(x)$ have their entries in $L^1(\mathbb{R})$. Moreover, $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(\breve{\mathcal{A}}, \breve{\mathcal{B}}, \breve{\mathcal{C}})$ are triplets of size compatible matrices such that $\mathcal{A}$ and $\breve{\mathcal{A}}$ only have eigenvalues with positive real parts.

Characterization Problem: main result

Let $x_0 \in \mathbb{R}$.

$$\boxed{q(x),\ r(x) \text{ in } L^1(\mathbb{R})} \quad \underset{\text{inverse problem}}{\overset{\text{direct problem}}{\rightleftarrows}} \quad \boxed{\begin{array}{c}\{\Omega(x+y),\ \breve{\Omega}(x+y)\}_{x,y\geq x_0}\\ \text{AND}:\\ \{\Xi(x+y),\ \breve{\Xi}(x+y)\}_{x,y\leq x_0}\end{array}}$$

with

a. For $x, y \geq x_0$,
$\Omega(x+y) = \rho(x+y) + Ce^{-(x+y)A}B,\ \breve{\Omega}(x+y) = \breve{\rho}(x+y) + \breve{C}e^{-(x+y)\breve{A}}\breve{B};$

b. For $x \geq x_0$ the Marchenko integral equations are uniquely solvable;

c. For $x, y \leq x_0$,
$\Xi(x+y) = \ell(x+y) + \mathcal{C}e^{(x+y)\mathcal{A}}\mathcal{B},\ \breve{\Xi}(x+y) = \breve{\ell}(x+y) + \breve{\mathcal{C}}e^{(x+y)\breve{\mathcal{A}}}\breve{\mathcal{B}};$

d. For $x \leq x_0$ the Marchenko integral equations are uniquely solvable.

Characterization Problem: The focusing case

In the FOCUSING CASE

$$r(x) = +q(x)^\dagger, \qquad \breve{\Omega}(x+y) = -\Omega(x+y)^\dagger,$$

and the Marchenko equations are always uniquely solvable as proved in

- C. van der Mee, *Direct and inverse scattering for skewselfadjoint Hamiltonian systems*, in: *Current Trends in Operator Theory and its Applications*, pp. 407–439. (2004)

Let $x_0 \in \mathbb{R}$.

$$\boxed{q(x) \text{in } L^1(\mathbb{R})} \quad \underset{\text{inverse problem}}{\overset{\text{direct problem}}{\rightleftarrows}} \quad \boxed{\begin{array}{c}\Omega(x+y)_{x,y\geq x_0} \\ \text{AND :} \\ \Xi(x+y)_{x,y\leq x_0}\end{array}}$$

a. For $x, y \geq x_0$, $\Omega(x+y) = \rho(x+y) + Ce^{-(x+y)A}B$;

b. For $x, y \leq x_0$, $\Xi(x+y) = \ell(x+y) + \mathcal{C}e^{(x+y)\mathcal{A}}\mathcal{B}$;

Characterization Problem: Sketch of the proof

For $x \geq x_0$ ($x_0 \in \mathbb{R}$), we have to prove that $q(x)$, $r(x) \in L^1(\mathbb{R})$ implies that the Marchenko equations are uniquely solvable with the kernel in the appropriate form.

- **Step 1.** Suppose there are no bound states. The potentials pair $q(x)$, $r(x) \in L^1(\mathbb{R})$ lead to the Marchenko equations intoduced before (see reference in the preceding page), where the integral kernels are given by

$$\Omega(x+y) = \rho(x+y), \quad \breve{\Omega}(x+y) = \breve{\rho}(x+y)$$

  with $\rho(x+y), \breve{\rho}(x+y) \in L^1(\mathbb{R})$.
- **Step 2.** Adding bound states. Applying a Darboux transformation (i.e., adding a finite number of bound states at the integral kernel), the corresponding Marchenko equations remain uniquely solvable as proved in:

  T. Aktosun and C. van der Mee, *A unified approach to Darboux transformations*, Inverse Problems **25**, 105003 (2009), 22 pp.

Characterization Problem: Sketch of the proof

The viceversa is not trivial and very technical.

Step one: Prove the following

Lemma (Right L^1-tail potentials) *Suppose the Marchenko equations are uniquely solvable for $x \geq x_0$ and have Marchenko kernels expressed in terms of the triplet matrices, where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ and the matrices A and $\breve{A}$ only have eigenvalues with positive real parts. Then there exists $x_1 \geq x_0$ such that the potential pair $\{q(x), r(x)\}$ obtained by solving the Marchenko equations*

$$\int_{x_1}^{\infty} dx \, (\|q(x)\| + \|r(x)\|) < +\infty.$$

The proof follows by contraction argument.

Characterization Problem: Sketch of the proof

Step two:

Theorem (Half-line potentials) Suppose the Marchenko equations are uniquely solvable for $x \geq x_0$ and have Marchenko kernels of the "appropriate" form, where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$ and the matrices A and $\breve{A}$ only have eigenvalues with positive real parts. Then the potential pair $\{q(x), r(x)\}$ obtained by solving the Marchenko equations satisfies

$$\int_{x_0}^{\infty} dx \, (\|q(x)\| + \|r(x)\|) < +\infty.$$

It is enough to prove that $\int_{x_0}^{x_1} dy \, (\|q(y)\| + \|r(y)\|) < +\infty$.

Characterization Problem: Sketch of the proof

It is not difficult to arrive at the following equation

$$r(w) = 2\,\overline{K}^{(\mathrm{dn})}(x, 2w - x) - 2\int_x^w dz\, r(z)\overline{K}^{(\mathrm{up})}(z, 2w - z).$$

fom which

$$\int_{x_0}^{x_1} dw\, r(w) = 2\int_{x_0}^{x_1} dw \overline{K}^{(\mathrm{dn})}(x, 2w - x) - 2\int_{x_0}^{x_1} dw \int_x^w dz\, r(z)\overline{K}^{(\mathrm{up})}(z, 2w - z).$$

Suppose that $\overline{K}^{(\mathrm{up})}(z, 2w - z)$ is a bounded continuous function in (z, w), where $x_0 \leq z \leq w \leq x_1$, i.e., $\|\overline{K}^{(\mathrm{up})}(z, 2w - z)\| \leq \kappa$ for all such (z, w).
The Volterra operator

$$(\mathcal{L}r)(w) = 2\int_x^w dz\, r(z)\overline{K}^{(\mathrm{up})}(z, 2w - z)$$

has a zero spectral radius on $L^1(\mathbb{R})$ and therefore $\int_{x_0}^{x_1} dy\, \|r(y)\| < +\infty$.
In the general case the idea is to approximate $(\mathcal{L}r)(w)$ by a sequence of integral operators $\mathcal{L}_n$ of the same type such that $\|\mathcal{L}_n - \mathcal{L}\| \to 0$ as $n \to +\infty$ in the operator norm on $L^1((x_0, x_1); \mathbb{C}^{n\times m})$. (Recall The limit of a sequence of Volterra operators with respect to the operator norm is a Volterra operator.)

Future works on the characterization problem

- Extend the results obtained so far in such a way to include the evolution in time of the scattering data;
- Solve the characterization problem for scattering data in $L^2(\mathbb{R})$;
- Obtain similar results for AKNS system with non vanishing potentials.

Inverse problem: how to reconstruct the potential in the reflectionless case

By inserting

$$\Omega(y) = Ce^{-yA}B, \quad \breve{\Omega}(y) = \breve{C}e^{-y\breve{A}}\breve{B}$$

in the (uncoupled) Marchenko equation, we have

$$K^{up}(x,y) + \breve{C}e^{-(x+y)\breve{A}}\breve{B} + \int_x^\infty dz \int_x^\infty ds\, K^{up}(x,s)Ce^{-(s+z)A}B\breve{C}e^{-(z+y)\breve{A}}\breve{B} = 0.$$

Looking for solution of the form $K^{up}(x,y) = H(x)e^{-y\breve{A}}\breve{B}$ and defining

$$Q(x) = \int_x^\infty dz\, e^{-zA}B\breve{C}e^{-z\breve{A}}, \qquad N(x) = \int_x^\infty ds\, e^{-s\breve{A}}\breve{B}Ce^{-sA},$$

we arrive at

$$K^{up}(x,y) = -\breve{C}e^{-x\breve{A}}\big(I - N(x)Q(x)\big)^{-1}e^{-y\breve{A}}\breve{B}.$$

Inverse problem: how to reconstruct the potential in the reflectionless case

Recalling the relation $q(x) = -2K^{up}(x,x)$ we get

$$q(x) = 2\breve{C}e^{-x\breve{A}}\big(I - N(x)Q(x)\big)^{-1}e^{-x\breve{A}}\breve{B}.$$

In a similar way, we can construct also the potential $r(x)$

$$r(x) = 2Ce^{-xA}\big(I - Q(x)N(x)\big)^{-1}e^{-xA}B.$$

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Thank you for your attention!!!