Camassa - Holm eq. and its generalizations

$$(1) \quad u_t + K(u^m)_x - (u^n)_{xxt} = \left[\frac{((u^n)_x)^2}{2} + u^n (u^n)_{xx} \right]_x$$

$K = \text{const} > 0$, $m, n \in \mathbb{N}$

standard Camassa-Holm: $n=1$, $m=2$, $K = 3/2$

$u = \varphi(x - ct)$, $c = \text{const} > 0$, $\varphi(0) = \bar{c}$ - travelling wave

Assume: $\varphi(\infty) = \varphi'(\infty) = \varphi''(\infty) = 0$

(A) $c^{\frac{n-m+1}{n}} > K \cdot \frac{n+1}{m+n}$, $\varphi(0) = \bar{c} = c^{\frac{1}{n}}$

Th. 1. 1) Let either $m=2$, $n=1$, $K = 3/2$ or $m=-1$, $n=-2$, $K=3$. The (1) possesses a peakon but not compact travelling wave solution

2) $n=1$, $m=2$, $0 < K < 3/2$. Then (1) has a peakon-cuspon type solution forming a cusp type singularity at the point $(0, c)$, i.e. $\varphi(s)$ has a cusp type singularity at $s=0$, $\varphi(0) = c$.

3) Let $n > 3$, $m > 1$ and (A) holds. Then (1) possesses a compact travelling wave peakon-cuspon solution developing a cusp type singularity at $(0, c^{1/n})$, i.e. $\varphi(s) \sim c^{\frac{1}{n}} - \text{const}\,|s|^{2/3}$, $s \to 0$, $\text{const} > 0$.

The solutions here are even: $\varphi(-s) = \varphi(s)$

Exercise: Study the case $1 \le n \le 3$, $m \ge 1$.

Proof: Put $s = x - ct$, $\underline{\varphi(s) \text{ even}}$. In the case of peakons we assume $\varphi(\infty) = \varphi'(\infty) = \varphi''(\infty) = 0$, while in the case of compact travelling waves $\varphi(s)$ is compactly supported.

$\varphi(0) = \bar{c} > 0 \Rightarrow$

$$c(\varphi^n)''' - c\varphi' + K(\varphi^m)' = \left[\frac{((\varphi^n)')^2}{2} + \varphi^n(\varphi^n)''\right]'$$

$$\Rightarrow c(\varphi^n)'' - c\varphi + K\varphi^m = \frac{((\varphi^n)')^2}{2} + \varphi^n(\varphi^n)'' + C_1$$

put $C_1 = 0$ and suppose $\varphi \geq 0$ everywhere

Let $z(s) = \varphi^n(s) \Leftrightarrow$

$$-c\, z^{\frac{1}{n}}(s) + K z^{\frac{m}{n}}(s) + c z''(s) = \frac{z'^2}{2} + z z''$$

$\downarrow$ multiply both sides by z' and integrating w.r.to s we get

$$-c\frac{z^{1+\frac{1}{n}}}{1+\frac{1}{n}} + K\frac{z^{\frac{m}{n}+1}}{\frac{m}{n}+1} + \frac{c}{2}z'^2 = \frac{z z'^2}{2} + C_2$$

$C_2 = 0 \Rightarrow$

$$(z')^2(c - z) = c_1 z^{1+\frac{1}{n}} - K_1 z^{\frac{m}{n}+1},$$

where $c_1 = \dfrac{2c}{1+\frac{1}{n}}$, $K_1 = \dfrac{2K}{\frac{m}{n}+1}$

Going back to $\varphi = z^{\frac{1}{n}}$ we have

$$\otimes \quad n^2\varphi^{2(n-1)}(\varphi')^2(c - \varphi^n) = c_1\varphi^{n+1} - K_1\varphi^{m+n}, \quad \varphi(0) = \bar{c} > 0$$

Thus,

$$n^2 \varphi^{2(n-1)}\left[(\varphi')^2(c-\varphi^n) - \frac{c_1}{n^2}\varphi^{3-n} + \frac{K_1}{n^2}\varphi^{m+2-n}\right] = 0$$

Put $C_2 = \frac{c_1}{n^2}$, $K_2 = \frac{K_1}{n^2}$ we get

$$\boxed{\varphi'^2(c-\varphi^n) = C_2\varphi^{3-n} - K_2\varphi^{m+2-n}} = 0$$

ODE separate variables, i.e.

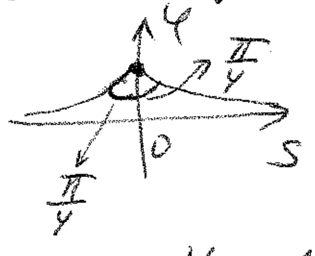
$$\varphi'^2(c-\varphi^n) = \begin{cases} \varphi^{3-n}\left(C_2 - K_2\varphi^{m-1}\right) & \text{or} \\ \varphi^{m+2-n}\left(-K_2 + C_2\varphi^{1-m}\right) \end{cases}$$

$\varphi(0) = \bar{c} > 0$

(*) In the case of classical Camassa-Holm $m=2$, $n=1$ we assume $\bar{c} = c$, $c = C_2$, $K_2 = 1$, i.e. $K = \frac{3}{2}$

$\Rightarrow \varphi'^2 = \varphi^2$ if $\varphi \not\equiv c$

Thus $\varphi(s) = c e^{-|s|}$ $\rightarrow$ peakon but non-compact travelling wave



peakon with opening $\frac{\pi}{2}$

(**) Suppose $n = m-1$, $c = C_2$ $K_2 = 1$ $\Rightarrow$

$\Rightarrow$ $n_1 = 1$, $m_1 = 2$, $K = \frac{3}{2}$ or $n_2 = -2$, $m_2 = -1$, $K = 3$ $\Rightarrow$

$$\Rightarrow (\varphi')^2 = \varphi^5 \Rightarrow \varphi(s) = \left(\tfrac{3}{2}|s| + \bar{c}^{-3/2}\right)^{-2/3}$$

(***) Let $n=1$, $m=2$, $K_2 < 1$, i.e. $0 < K < \frac{3}{2}$

$c = \bar{c} = C_2 = \varphi(0)$

Then $(\varphi'^2)(c-\varphi) = \varphi^2(c_{\bullet} - K_2\varphi)$, $c - K_2\varphi > c - \varphi \ge 0$,

$\varphi \ge 0$. The corresponding peakon is given by

$$e^{-|s|} = \frac{\left(1-\sqrt{\frac{c-\varphi}{c-K_2\varphi}}\right)\left(1+\sqrt{\frac{c-\varphi}{\frac{c}{K_2}-\varphi}}\right)^{\sqrt{1/K_2}}}{\left(1+\sqrt{\frac{c-\varphi}{c-K_2\varphi}}\right)\left(1-\sqrt{\frac{c-\varphi}{\frac{c}{K_2}-\varphi}}\right)^{\sqrt{1/K_2}}}$$

$\downarrow$ cusp singularity at $(0, \bar{c})$

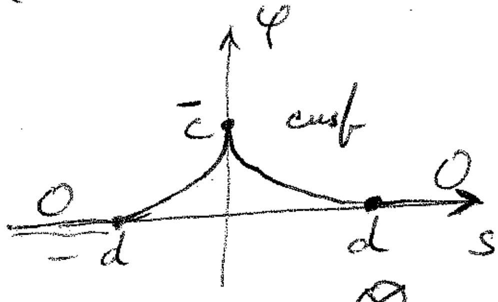
In the case $\varphi(0) = \bar{c} = c^{\frac{1}{n}}$, $c_2 - K_2 c^{\frac{m-1}{n}} > 0$

we can construct the following solution $\varphi(s)$, $|s| \le d$

$\varphi \sim c^{\frac{1}{n}} - \text{const}\,|s|^{2/3}$, $s \to 0 \Rightarrow \varphi' \sim -\text{const}\,|s|^{-1/3}$

$\varphi \sim \text{const}\,(d-|s|)^{\frac{2}{n-1}}$, near $|s|=d$, $|s|<d$

we continue φ as 0 for $|s|>d$



$\frac{2}{n-1} - 1 = \frac{2-n+1}{n-1}$

$2 + \frac{3-n}{n-1} = \frac{2n-2+3-n}{n-1}$

We know that $\otimes$ $\varphi'^2(c-\varphi^n) \sim \text{const}\,|s|^{-2/3}$

near $(s=0)$ $\times \left[c - (c + O(|s|^{2/3}))\right] \Rightarrow$

$\Rightarrow$ i.e. left hand side does not have singularity at $s=0$

right hand side: $c_1\varphi^{n+1} - K_1\varphi^{m+n} \to$

no singularities at $s=0$, Eq $\otimes$ is satisfied

near $|s|=d \to$ left hand side:

$\otimes$ $\varphi^{2(n-1)} \cdot \varphi'^2 = (\varphi' \cdot \varphi^{n-1})^2 \sim \text{const}\left[(d-|s|)^2 \cdot (d-|s|)^{\frac{3-n}{n-1}}\right]^2 =$

$= \text{const}\,(d-|s|)^{\frac{n+1}{n-1}\cdot 2}$

Fornberg – Whitham equation – travelling waves

$$u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_x u_{xx}$$

$$u = \varphi(x-ct), \quad \xi = x - ct \Rightarrow$$

$$\Rightarrow -c\varphi' + c\varphi''' + \varphi' + \varphi\varphi' = \varphi\varphi''' + 3\varphi'\varphi''$$

Integrate w.r. to ξ, having in mind that

$$(\varphi\varphi'')' = \varphi\varphi''' + \varphi'\varphi'' \Rightarrow$$

$$c\varphi'' + \varphi(1-c) + \tfrac{1}{2}\varphi^2 = \varphi\varphi'' + \varphi'^2 - g, \quad g = \text{const}$$

$$\Rightarrow \varphi''(\varphi - c) = \tfrac{1}{2}\varphi^2 + \varphi(1-c) - \varphi'^2 + g$$

Change: $\varphi' = p(\varphi) \Rightarrow \varphi'' = p\dfrac{dp}{d\varphi} = \dfrac{1}{2}\dfrac{d}{d\varphi}(p^2)$

and substitution $q(\varphi) = p^2(\varphi) \Rightarrow$

$$(\varphi - c)\frac{dq}{d\varphi} = \varphi^2 + 2\varphi(1-c) - 2q + 2g \rightarrow$$

linear ODE $\Rightarrow$

$$\Rightarrow q(\varphi) = \frac{1}{(\varphi-c)^2}\left[\bar{C}_1 + \int \underbrace{(\varphi^2 + 2\varphi(1-c) + 2g)}(\varphi - c)\,d\varphi\right]$$

$\bar{C}_1 = \text{const}$. Put $T_2(\varphi) = \varphi^2 + 2\varphi(1-c) + 2g \Rightarrow$

$$\Rightarrow q(\varphi) = \frac{1}{(\varphi-c)^2}\left[\bar{C}_1 + \int T_2(\varphi)(\varphi - c)\,d\varphi\right], \quad q(\varphi) \geq 0$$

After two integrations by parts we get

$$q(\varphi) = \frac{\overline{C}_1}{(c-\varphi)^2} + \widetilde{\Gamma}_2(\varphi),$$ where the quadratic

polynomial $\widetilde{\Gamma}_2(\varphi) = \frac{1}{2}\Gamma_2(\varphi) - \frac{1}{6}\Gamma_2'(\varphi - c) + \frac{1}{12}(\varphi - c)^2 =$

$$= \frac{1}{4}\varphi^2 + \frac{4-3c}{6}\varphi + g + \frac{c(4-3c)}{12}.$$

Thus

$$\varphi'^2(\xi) = \frac{\overline{C}_1 + (\varphi - c)^2\,\widetilde{\Gamma}_2(\varphi)}{(\varphi - c)^2} \Leftrightarrow$$

$$(\varphi - c)^2\,\varphi'^2 = \underbrace{\overline{C}_1 + \Gamma_2(\varphi - c)^2}_{\geq 0 \text{ (Assume)}}, \quad \varphi \not\equiv c$$

Then we consider two different cases

(A1) $\overline{C}_1 = 0 \Rightarrow \varphi'^2 = \widetilde{\Gamma}_2(\varphi)$

(A2) $\overline{C}_1 \neq 0$

Case (A1). Denote by $\widetilde{\Delta}(c,g) = \frac{(c-1)^2}{2} - \frac{1}{18} - g$ the discriminant of $\widetilde{\Gamma}_2(\varphi)$. Then 3 cases appear:

(A1)(a): $\widetilde{\Delta} > 0$, i.e. $g < \frac{(c-1)^2}{2} - \frac{1}{18}$

(A1)(b) $\widetilde{\Delta} < 0$, i.e. $g > \frac{(c-1)^2}{2} - \frac{1}{18}$

(A1)c : $\widetilde{\Delta} = 0$,

Put $X(\varphi) = a\varphi^2 + b\varphi + \bar{c}$, $a>0$, $\Delta = b^2 - 4a\bar{c}$

Then

$$\int \frac{d\varphi}{X(\varphi)} = \frac{1}{\sqrt{a}} \log \left|2\sqrt{a}\sqrt{X(\varphi)} + 2a\varphi + b\right| = \frac{1}{\sqrt{A}} \log |B|.$$

In the case (A1)(a) the eq. $X(\varphi)=0$ has 2 real distinct roots $\alpha < \beta$, $B = 2\sqrt{a}\sqrt{X(\varphi)} + 2a\varphi + b$ for $\beta < \varphi$, while $B<0$ for $\varphi < \alpha$.

In the case (A1)(b) the function $B(\varphi) > 0$, $\forall \varphi$, while in the case (A1)(c) (double root)

$$\int \frac{d\varphi}{\sqrt{X(\varphi)}} = \frac{1}{\sqrt{a}} \begin{cases} \log(2a\varphi + b), & 2a\varphi + b > 0 \\ -\log|2a\varphi + b|, & 2a\varphi + b < 0. \end{cases}$$

Moreover, in the case (A1)(b) $\int \frac{d\varphi}{\sqrt{X(\varphi)}} = \frac{1}{\sqrt{a}} \operatorname{Arcsh} \frac{2a\varphi + b}{\sqrt{4a\bar{c} - b^2}}$, Arcsh being the inverse function of $\operatorname{sh} W$, $\frac{d}{dz} \operatorname{Arcsh} z = \frac{1}{\sqrt{1+z^2}}$

Conclusion: If (A1)(b) holds, then

$$\varphi = \pm 2\sqrt{-\tilde{\Delta}}\ \operatorname{sh} \frac{\xi + C_2}{2} + c - \frac{4}{3} \rightarrow$$

unbounded solution $|\varphi(\pm\infty)| = \infty$.

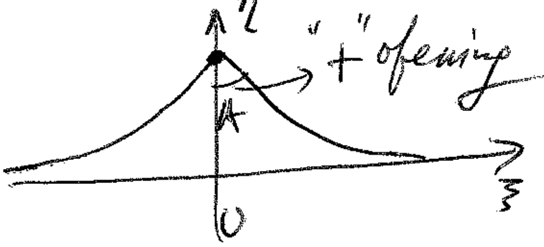
Let (A1)(a) holds $\Rightarrow \varphi'^2 = \tilde{r}_2 = \left(\frac{\varphi}{2} + \frac{\gamma - 3c}{6}\right)^2$,

double root

Translation: $\varphi = \frac{3c-\gamma}{6} + \eta \Rightarrow \eta'^2 = \frac{1}{4}\eta^2 \Rightarrow$

$\Rightarrow \eta' = \pm\left|\frac{\eta}{2}\right|$. If $\eta(0)=0 \Rightarrow \eta \equiv 0$.

If $\eta(0) \neq 0 \Rightarrow \eta = A e^{-\frac{|\xi|}{2}} \rightarrow$ peakon-solition



$\Rightarrow \boxed{\varphi = A e^{-\frac{|\xi|}{2}} + c - \frac{\gamma}{3}}$

Assume $c = \frac{\gamma}{3} \Rightarrow \varphi = A e^{-1/2\left|x - \frac{\gamma}{3}t\right|} \rightarrow$

Fornberg-Whitham solution

Conclusion: (A1). There exist unbounded solution in the case (A1)(a), (A1)(b). Peakon-solition exists only for $\tilde{\Delta} = 0$.

Hunter – Saxton equation

(1) $(u_t + u u_x)_x = \frac{1}{2} u_x^2, \quad t \geq 0 \quad \Rightarrow \quad \boxed{u_t + u_x u = \int_{-\infty}^{x} \frac{1}{2} u_x^2(t,\lambda)\,d\lambda} = g(t,x)$

$u(0,x) = F(x) \in C^2(R^1), \quad \int_{-\infty}^{\infty} F'^2\,dx < \infty,$

$F(x) \xrightarrow[x \to \infty]{} 0$

Conservation law: $(1) \Rightarrow (u_x^2)_t + (u u_x^2)_x = 0, \quad t \geq 0$

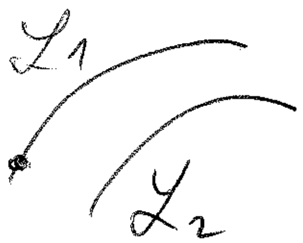
Characteristic curve of (1):

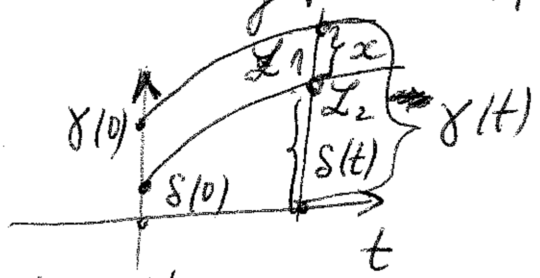
$\mathcal{L}: \boxed{x = x(t,\alpha)}$, shortly $x = \gamma(t)$ satisfies

$$\left| \begin{aligned} &\frac{d\gamma}{dt} = u(t,\gamma(t)) \\ &\gamma(0) = \alpha \in R^1 \end{aligned} \right. \quad \Rightarrow \quad \frac{d}{dt} u(t,\gamma(t)) = u_t + u_x \cdot \underset{u}{\underset{\parallel}{\gamma'}} =$$

$$= u_t + u u_x = \frac{1}{2}\int_{-\infty}^{\gamma(t)} u_x^2(t,y)\,dy \equiv g(t,\gamma(t))$$

Suppose $F' \geq 0$. Let $\mathcal{L}_1: x = \gamma(t)$, $\mathcal{L}_2: x = \delta(t)$ be two characteristics globaly defined for $t \geq 0$





We integrate the conservation law w.r.to x from $\delta(t)$ till $\gamma(t)$:

$$0 = \int_{\delta(t)}^{\gamma(t)} \left(u_x^2(t,x)\right)_t dx + u(t,\gamma(t))\,u_x^2(t,\gamma(t)) - u(t,\delta(t)) \times u_x^2(t,\delta(t))$$

i.e. $\frac{\partial}{\partial t} \int_{\delta(t)}^{\gamma(t)} u_x^2(t,x)\,dx =$

$$= \int_{\delta(t)}^{\gamma(t)} \left(u_x^2(t,x)\right)_t dx + u_x^2(t,\gamma(t))\cdot\underbrace{\gamma'}_{u(t,\gamma)} - u_x^2(t,\delta(t))\underbrace{\delta'}_{u(t,\delta)}$$

$= 0$

$$\Rightarrow \int_{\delta(t)}^{\gamma(t)} u_x^2(t,x)\,dx = \text{const} = \int_{\delta(0)}^{\gamma(0)} u_x^2(0,x)\,dx = \int_{\delta(0)}^{\gamma(0)} F'^2\,dx < \infty$$

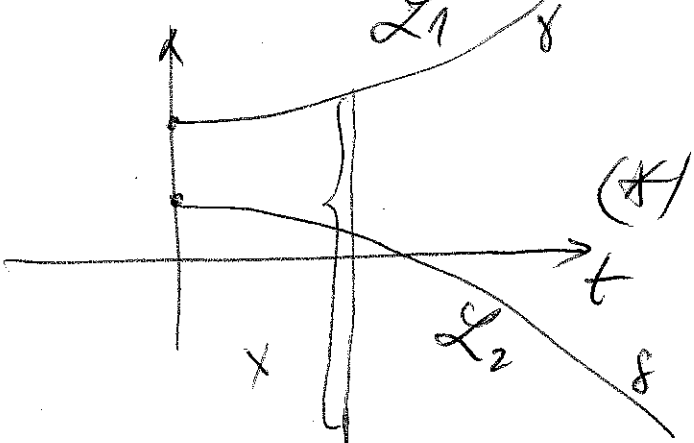
Suppose that $\gamma(0) = \delta(0) \Rightarrow \gamma(t) = \delta(t),\ \forall t \geq 0$

$\Rightarrow$ global uniqueness for the characteristics

If $\delta(0) < \gamma(0) \Rightarrow \delta(t) < \gamma(t),\ \forall t > 0$.

$\mathcal{L}_1, \mathcal{L}_2$ are not crossing smooth curves

Suppose now that $\gamma(t) \xrightarrow[t\to\infty]{} \infty,\ \delta(t) \xrightarrow[t\to\infty]{} -\infty$



(*) then $\int_{-\infty}^{\infty} u_x^2(t,x)\,dx = \text{const}$

$\downarrow$

conservation law

Proof of (*): Consider the identity

$$\frac{\partial}{\partial t} \int_{-\infty}^{\gamma(t)} u_x^2(t,x)\,dx = \int_{-\infty}^{\gamma(t)} \left(u_x^2(t,x)\right)_t dx + \underbrace{\gamma'}_{u(t,\gamma)} u_x^2(t,\gamma)$$

Assume $\int_{-\infty}^{\infty} u_x^2(t,x)\,dx\ \infty \Rightarrow u_x^2(t,x) \xrightarrow[x\to\pm\infty]{} 0$

Integrating $(u_x^2)_t + (u u_x^2)_x = 0 \;;\; -\infty < x < \gamma(t)$

we get $\int_{-\infty}^{\gamma(t)} (u_x^2)_t(t,x)\,dx + u(t,\gamma)u_x^2(t,\gamma) -$

$$-\lim_{x\to-\infty} u(t,x)u_x^2(t,x) = 0$$

Assume: $\lim_{x\to-\infty} u(t,x)u_x^2(t,x) = 0 \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\gamma(t)} u_x^2(t,x)\,dx = 0 \quad \Rightarrow$$
$(t \geq 0)$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\gamma(t)} u_x^2(t,x)\,dx = \frac{1}{2}\int_{-\infty}^{\gamma(0)} u_x^2(0,x)\,dx =$$

$$= \frac{1}{2}\int_{-\infty}^{\alpha} F'^2(x)\,dx = \text{const}, \text{ i.e.}$$

$$\boxed{g(t,\gamma(t)) = g(0,\alpha)}, \quad \forall t \geq 0$$

$$\gamma(0) = \alpha$$

Conclusion: $\frac{d\gamma}{dt} = u(t,\gamma(t)),\ \gamma(0) = \alpha$

$$\frac{d}{dt}u(t,\gamma(t)) = g(t,\gamma(t)) \equiv g(0,\alpha)$$

$$u\big|_{t=0} = u(0,\underbrace{\gamma(0)}_{\alpha}) = F(\alpha)$$

Therefore:

$$u(t, \underbrace{\gamma(t)}_{x}) = F(\alpha) + t\, g(0,\alpha) = F(\alpha) + \frac{t}{2}\int_{-\infty}^{\alpha} F'^2(x)\,dx$$

$$x = \gamma(t) = \alpha + t F(\alpha) + \frac{t^2}{4}\int_{-\infty}^{\alpha} F'^2(x)\,dx$$

So: $(t,\alpha) \to (t,x)$;

$$\underset{(t,\alpha)}{x} = \alpha + t\underbrace{F(\alpha)}_{bdd} + \frac{t^2}{4}\underbrace{\int_{-\infty}^{\alpha} F'^2(x)\,dx}_{bdd}$$

$t \to$ parameter — smooth

$x \Leftrightarrow \alpha$ we want to invert the mapping

$$\frac{\partial x}{\partial \alpha} = 1 + tF'(\alpha) + \frac{t^2}{4}F'^2(\alpha) =$$

$$= \left(1 + \frac{t}{2}F'(\alpha)\right)^2 > 0 \text{ for } F'(\alpha) \geq 0 \quad \forall \alpha \in \mathbb{R}^1$$

Let $F' \in L^\infty$, $\int_{-\infty}^{\infty} F'^2(x)\,dx < \infty$

Thus for each fixed $t \geq 0$: $\alpha \to \pm\infty \Rightarrow$

$\Rightarrow x \to \pm\infty$, i.e. the mapping is <u>onto</u> for each fixed $t \geq 0$. Implicit function th. with smooth parameter implies that there exists and inverse function $\alpha = \alpha(t,x) \in C^2(t \geq 0)$ of $x(t,\alpha)$

Thus: $\boxed{u(t,x) = F(\alpha(t,x)) + \frac{t}{2}\int_{-\infty}^{\alpha(t,x)} F'^2(s)\,ds}$